(Marks)

(10) 1. (a) Give an infinite series in powers of x for the function $f(x) = \int_0^x \sin t^2 \, dt$.

(b) What is the radius of convergence for this power series?

(c) Use the first 3 terms of this series to estimate the value of $f(1)$. How precise is your estimate?

(5) 2. Given $f(x) = x\,e^x$:

(a) Find the Maclaurin series for $f(x)$ and for $f'(x)$.

(b) Use your answer to 2(a) to find the exact value (*i.e.* not a decimal approximation) of the series

$$1 + 2 + \frac{3}{2!} + \frac{4}{3!} + \frac{5}{4!} + \cdots$$

(10) 3. Suppose that a curve $\mathcal{C}$ given by parametric equations in t passes through the origin at $t = 0$ and satisfies

$$\frac{dx}{dt} = 3\sin^2 t \cos t \quad \text{and} \quad \frac{dy}{dt} = -3\sin t \cos^2 t$$

(a) Find the parametric equations for $\mathcal{C}$, (*i.e.* for x, y) in terms of t.

(b) Find an equation for $\mathcal{C}$ (*i.e.* for x, y) in rectangular coordinates by eliminating the parameter in 3(a).

(c) Find the arc length of $\mathcal{C}$ from $t = 0$ to $t = \frac{\pi}{2}$.

(8) 4. (a) Sketch the graphs of $r = 3\cos\theta$ and $r = 1 + \cos\theta$ on the same axes.

(b) Find all their points of intersection.

(c) Find the area inside $r = 3\cos\theta$ and outside $r = 1 + \cos\theta$.

(5) 5. Convert the equation $\rho = 4\sin\phi\cos\theta$ into rectangular coordinates. Sketch and name the surface.

(8) 6. A curve is defined by $\boldsymbol{r}(t) = \langle t\cos t\,,\, t\sin t\rangle$. Find the velocity and acceleration vectors: $\boldsymbol{v}(t), \boldsymbol{a}(t)$. Find the unit tangent and unit normal vectors at $t = 0$, *i.e.* $\boldsymbol{T}(0), \boldsymbol{N}(0)$. Find the curvature $\kappa(0)$, *i.e.* at $t = 0$.

(6) 7. Given the function $f(x, y) = \sqrt{x^2 + 2y^2 - 1}$:

(a) Sketch the domain of f in the xy plane.

(b) Sketch and name the surface $z = f(x, y)$, clearly labelling the traces of the graph in the coordinate planes.

(12) 8. Given the level surface $\mathcal{S}$: $f(x, y, z) = x - y^3 - 2z^2 = 2$ and the point $P(-4, -2, 1)$, find:

(a) the equation of the tangent plane to $\mathcal{S}$ at the point P;

(b) the directional derivative of f at P in the direction of $\boldsymbol{v} = \langle 3, 6, -2\rangle$;

(c) the maximum rate of change in f at P;

(d) the equations of the tangent line at P to the curve of intersection of $\mathcal{S}$ and the plane $2x - 3y - z = -3$.

(4) 9. If $z = f(x, y)$ is implicitly defined by $\sin(x^2 z) + y\cos x = z\sec y$, find $\dfrac{\partial z}{\partial x}$.

(Marks)

(4) 10. If $f(u,v,w)$ is a differentiable function, and $p = f(x-y\,,\,y-z\,,\,z-x)$ show that

$$\frac{\partial p}{\partial x} + \frac{\partial p}{\partial y} + \frac{\partial p}{\partial z} = 0$$

(6) 11. Find and classify the critical points of $f(x,y) = 6x^2 - 2x^2y + 3y^2$.

(6) 12. Use Lagrange Multipliers to find the shortest distance from the point $(1,0,3)$ to the surface $z = x - 2y^2$.

(10) 13. Evaluate the following:

(a) $\displaystyle\iint_R \frac{y}{\sqrt{1+x^3}}\,dA$, where R is the triangular region with vertices $(0,0), (2,0), (2,2)$.

(b) $\displaystyle\int_0^1 \int_y^{\sqrt{2-y^2}} e^{x^2+y^2}\;dx\,dy$

(6) 14. Sketch the solid region $\mathcal{S}$ inside the sphere $x^2+y^2+z^2 = 4z$ and above the cone $z = \sqrt{x^2+y^2}$. Calculate its volume.