Calculus III — 201-DDB — Final Exam
13 December 2000

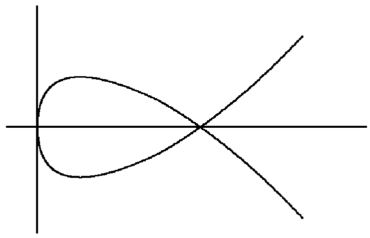
1. (a) Write a Taylor polynomial of degree 3 for $f(x) = \ln x$ about $c = 2$
   (b) Use part (a) to approximate $\ln(2.5)$
   (c) Use Taylor's formula for the Remainder to estimate the accuracy of your answer.

2. Given $f(x) = \displaystyle\int_0^x \frac{1-\cos t}{t}\,dt$
   (a) Find the Maclaurin series for $f(x)$
   (b) Use the result of (a) to approximate $f(0.5)$ correctly to 5 decimal places
   (c) Find $f^{(6)}(0)$.

3. (a) Use the Binomial series to expand $f(x) = \dfrac{1}{\sqrt{1-x^2}}$
   (b) Use part (a) to find the Maclaurin series for $\sin^{-1} x$
   (c) Use the first five terms in part (b) to approximate π

4. Let C be the curve having parametric equations

$$\begin{cases} x(t) = t^2 \\ y(t) = t^3 - 4t \end{cases}$$



   (a) Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$
   (b) Indicate the orientation on the graph of C
   (c) Set up the integral needed to find the area of the loop formed as $-2 \le t \le 2$

5. Find
   (a) the area under one arch of the cycloid
   (b) the length of one arch of the same cycloid

$$\begin{cases} x = a(t - \sin t) \\ y = a(1 - \cos t) \end{cases}$$

6. Find the points of intersection of the polar curves $r = \sin\theta$ and $r = \sin 2\theta$ and set up the integral(s) needed to find the area of the region common to both

7. Consider the space curve C defined by the vector equation $r(t) = \left\langle \sin t, \sin t, \sqrt{2}\cos t \right\rangle$
   (a) Sketch and describe the space curve C
   (b) Calculate the length of C on the interval $[0, \pi]$
   (c) Find $T(t)$, $N(t)$, a_T , a_N and $\kappa(t)$

8. Sketch the graphs of the following:
   (a) the surface $r^2 - z^2 = 4$
   (b) the surface $\rho\cos\phi = 4r$
   (c) the domain of $f(x,y) = \sqrt{x^2+y^2-1} + \ln(4 - x^2 - y^2)$
   (d) the level curve of $f(x,y) = \dfrac{2x}{x^2+y^2}$ corresponding to $c = 1$

9. Find and classify the critical points of $f(x,y) = x^2 + 2y^2 - x^2 y$

10. Let $z = f(x,y) = \sin(x^2 + y^2)$

    (a) Find the total differential dz

    (b) Use part (a) to approximate $\Delta z = f(Q) - f(P)$ where $P = (1,1)$ & $Q = (1.05, 0.95)$

11. Given that $u(x,y) = \dfrac{x^2}{x+y}$ verify that the conclusion of Clairaut's theorem holds *i.e.* $u_{xy} = u_{yx}$

12. Let $f(x,y,z) = 2x^2 + y^2 - z$

    (a) Show that the line

$$\begin{cases} x = -2 + t \\ y = t \\ z = -4 + 4t \end{cases}$$

    is tangent to the surface $f(x,y,z) = 0$ at a certain point.

    (b) Find the rate of change of $f(x,y,z)$ in the direction of the given line given by increasing t

13. Given $w = f(u,v)$; $u = \dfrac{1}{x} - \dfrac{1}{y}$ & $v = \dfrac{1}{x} - \dfrac{1}{z}$ Show that $x^2 \frac{\partial w}{\partial x} + y^2 \frac{\partial w}{\partial y} + z^2 \frac{\partial w}{\partial z} = 0$

14. Evaluate $\displaystyle\int_0^8 \int_{\sqrt[3]{x}}^2 \frac{1}{1+y^4}\, dy\, dx$

15. Find the volume of the solid cut out of the sphere $x^2 + y^2 + z^2 = 9$ by the cylinder $r = 3\sin\theta$

16. Express $\displaystyle\int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{x^2+y^2}^{4} dz\, dy\, dx$

    (a) in the form $\iiint_S dx\, dz\, dy$

    (b) in cylindrical &

    (c) in spherical coordinates

Answers to Cal 3 Final Exam (Dec. 2000)

1. a) $\ln x = \ln 2 + \frac{(x-2)}{2} - \frac{(x-2)^2}{8} + \frac{(x-2)^3}{24}$

b) $\ln(2.5) \approx 0.9171055$

c) $R_3(x) \leq 0.0009766$

2 i) $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^{2n}}{(2n)(2n)!}$ ii) $f(0.5) \approx 0.06185$ error≤ 0.00000362

iii) $f^{(6)}(0) = \frac{1}{6}$

3 a) $1 + \sum_{n=1}^{\infty}\frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2^n n!}x^{2n}$

b) $x + \sum_{n=1}^{\infty}\frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2^n n!}\frac{x^{2n+1}}{2n+1}$

c) 3.141511

4 i) $\frac{dy}{dx} = \frac{3t^2-4}{2t}$; $\frac{d^2y}{dx^2} = \frac{3t^2+4}{4t^3}$ $A = 2\int_{-2}^{0}(8t^2 - 2t^4)dt$

5 i) $A = 3\pi a^2$ *ii)* $s = 8a$

6 $(\frac{\sqrt{3}}{2}, \frac{\pi}{3})$ $(\frac{\sqrt{3}}{2}, \frac{2\pi}{3})$ $(0,0)$ $A = \int_0^{\pi/3}\sin^2\theta \, d\theta + \int_{\pi/3}^{\pi/2}\sin^2 2\theta \, d\theta$

7 a) circle b) $s = \sqrt{2}\,\pi$ c) $T(t) = \frac{1}{\sqrt{2}}(\cos t, \cos t, -\sqrt{2}\sin t)$;

$N(t) = -\frac{1}{\sqrt{2}}(\sin t, \sin t, \sqrt{2}\,\cos t)$; $a_T = 0$; $a_N = \sqrt{2}$; $K(t) = \frac{1}{\sqrt{2}}$

8 a) hyperboloid of 1 sheet b) top half of a cone c) circle centered at (1,0) & radius 1

d) region between a circle of radius 1 & a circle of radius 2

9 (0,0) a rel. min ; (2,1) and (-2,1) saddle points.

10 $dz = 2x\,\cos(x^2+y^2)\,dx + 2y\cos(x^2+y^2)\,dy$ $\Delta z \approx dz = 0$

11 $u_x = \frac{x^2+2xy}{(x+y)^2}$ $u_{xy} = \frac{-2xy}{(x+y)^3}$ $u_y = \frac{-x^2}{(x+y)^2}$ $u_{yx} = \frac{-2xy}{(x+y)^3}$

12 i) Point of intersection (0,2,4) ; $\nabla f(P) = (0,4,-1)$; $v = (1,1,4)$; $\nabla f \cdot v = 0$

ii) $D_v f = 0$

14 $\frac{\ln 17}{4}$

15 $V = 18\pi - 24$ (≈ 32.54)

16 i) $\int_{-2}^{2}\int_{y^2}^{4}\int_{-\sqrt{z-y^2}}^{\sqrt{z-y^2}} dx\;dz\;dy$

ii) $\int_0^{2\pi}\int_0^2\int_{r^2}^4 r\;dz\;dr\;d\theta$

iii) $\int_0^{2\pi}\int_0^{\tan^{-1}(\frac{1}{2})}\int_0^{4\sec\varphi}\rho^2\sin\varphi d\rho\,d\varphi\,d\theta + \int_0^{2\pi}\int_{\tan^{-1}(\frac{1}{2})}^{\pi/2}\int_0^{\csc\varphi\cot\varphi}\rho^2\sin\varphi\;d\rho\,d\varphi\,d\theta$