Final Examination Calculus III

Answers

1 $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$; $e^{-x^3} = \sum_{k=0}^{\infty} \frac{(-1)^k x^{3k}}{k!}$; $\int_0^{.5} e^{-x^3} = \sum_{k=0}^{\infty} \frac{(-1)^k (0.5)^{3k+1}}{k!\,(3k+1)}$

$|E_2| < \frac{(0.5)^{10}}{3!\,(10)} = 1.63 \times 10^{-5} < 5 \times 10^{-5}$; $\int_0^{.5} e^{-x^3} = \sum_{k=0}^{2} \frac{(-1)^k (0.5)^{3k+1}}{k!\,(3k+1)} = 0\ .484\,9$

2a $\frac{1}{2x-5} = \frac{-1}{5}\left(\frac{1}{1-\frac{2}{5}x}\right) = \frac{-1}{5}\sum_{k=0}^{\infty}\left(\frac{2x}{5}\right)^k = -\frac{1}{5} - \frac{2}{25}x - \frac{4}{125}x^2 - \frac{8}{625}x^3 + O\left(x^4\right)$

valid for $\left|\frac{2x}{5}\right| < 1$ i.e. $|x| < \frac{5}{2}$

b $e^x \sin x = \left(\sum_{k=0}^{\infty}\frac{x^k}{k!}\right)\left(\sum_{k=0}^{\infty}\frac{(-1)^k x^{2k+1}}{(2k+1)!}\right) = \left(1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4\right)\left(x - \frac{1}{6}x^3 + \frac{1}{120}x^5\right)$

$= x + x^2 + \frac{1}{3}x^3 - \frac{1}{30}x^5 + O\left(x^6\right)$

3a $x - \sin x = x - \sum_{k=0}^{\infty}\frac{(-1)^k x^{2k+1}}{(2k+1)!} = \sum_{k=1}^{\infty}\frac{(-1)^{k+1} x^{2k+1}}{(2k+1)!}$ $\quad x \in R$;

$x^3 \cos x = x^3 \sum_{k=0}^{\infty}\frac{(-1)^k x^{2k}}{(2k)!} = \sum_{k=0}^{\infty}\frac{(-1)^k x^{2k+3}}{(2k)!}$ $x \in R$

b $\lim_{x->0} \frac{\sum_{k=1}^{\infty}\frac{(-1)^{k+1} x^{2k+1}}{(2k+1)!}}{\sum_{k=0}^{\infty}\frac{(-1)^k x^{2k+3}}{(2k)!}} = \lim_{x->0}\left[\frac{\frac{1}{6}x^3 - \frac{1}{120}x^5 + O\left(x^7\right)}{x^3 - \frac{1}{2}x^5 + O\left(x^7\right)}\right] = \frac{1}{6}$

4 $\sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \frac{5}{128}x^4 + O\left(x^5\right)$

$\sqrt{105} = 10\sqrt{1 + \frac{1}{20}} = 10\left[1 + \frac{1}{40} - \frac{1}{3200} + \frac{1}{128000} - + ...\right] = 10\left[1 + \frac{1}{40} + \sum_{k=2}^{\infty}\frac{(-1)^k (1.3.5....(2k-3))}{40^k\,(k!)}\right]$

$E_2 < 10\left(\frac{3}{40^3(6)}\right) = 7.812\,5 \times 10^{-5} < 10^{-4}$ $\qquad \sqrt{105} = 10\left[1 + \frac{1}{40} - \frac{1}{3200}\right] \approx 10.246\,9$

5b $x = t,\ y = 2,\ z = 2t,\ t \in R$ **c** $\sqrt{5}\pi$ **d** $0,\ 2,\ \frac{1}{\sqrt{5}}(1, -2\sin t, 2\cos t),\ (0, -\cos t, -\sin t)$

e $\frac{2}{5}$

6a $\frac{dx}{dt} = 3t(t-2) \quad \frac{dy}{dt} = 2t\,(t-2)$ The curve is not smooth when both $\frac{dx}{dt} = \frac{dy}{dt} = 0 \Leftrightarrow$ $t = 2 \Rightarrow (-4,-4)$

b $\frac{dy}{dx} = \frac{2t(t-2)}{3t(t-2)} \quad \lim_{t->2} \frac{dy}{dx} \neq \pm\infty$ **c** $\int_3^4 \sqrt{9t^2(t-2)^2 + 4\,(t-2)^2}\,dt = \int_3^4 (t-2)\sqrt{9t^2+4}\,dt$

d $\frac{d^2y}{dx^2} = \frac{3}{4\,(t-2)} > 0$ if $t > 2$

7 $2 \cdot \frac{1}{2} \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \left(9\sin^2\theta - \left(8 - 4\sqrt{2}\sin\theta + \sin^2\theta\right)\right) d\theta = 6 - \pi = 2.858\,41$

8a $\nabla f(P_0) = (-4, 2, -4)$; $2x - y + z = 7$ **8b** $\frac{-2}{\sqrt{6}}$ **c** $6,\ (-4,2,-4)$

9a $\frac{\partial z}{\partial r} = \frac{\partial f}{\partial x}\cos\theta + \frac{\partial f}{\partial x}\sin\theta$; $\frac{\partial z}{\partial \theta} = \frac{\partial f}{\partial x}(-r\sin\theta) + \frac{\partial f}{\partial x}(r\cos\theta)$

b $\left(\frac{\partial z}{\partial \theta}\right)^2 = \left(\frac{\partial f}{\partial x}(-r\sin\theta) + \frac{\partial f}{\partial x}(r\cos\theta)\right)^2 = r^2\sin^2\theta\left(\frac{\partial f}{\partial x}\right)^2 - 2r^2\sin\theta\cos\theta\frac{\partial f}{\partial x}\frac{\partial f}{\partial y} +$ $r^2\cos^2\theta\left(\frac{\partial f}{\partial y}\right)^2$

$\left(\frac{\partial z}{\partial r}\right)^2 = \left(\frac{\partial f}{\partial x}\cos\theta + \frac{\partial f}{\partial x}\sin\theta\right)^2 = \cos^2\theta\left(\frac{\partial f}{\partial x}\right)^2 + 2\sin\theta\cos\theta\frac{\partial f}{\partial x}\frac{\partial f}{\partial y} + \sin^2\theta\left(\frac{\partial f}{\partial y}\right)^2$

10 saddle point at $(0,3)$, relative minima at $(\pm 2, 3)$ **11** max $f(3,3,4) = 28$, min $f(-3,-3,-4) = -42$

12a(ii) $\int_0^3 \int_{-1}^1 1 - x^2\,dx\,dy$ **b(ii)** $\int_0^{2\pi} \int_0^{\frac{\pi}{4}} \int_0^{\cos\phi} \rho^2 \sin\phi\,d\rho\,d\phi\,d\theta$

13a $\int_0^{\frac{\pi}{4}} \int_0^2 e^{r^2}\,r\,dr\,d\theta = \frac{\pi}{8}(e^4 - 1)$ **b** $\int_0^4 \int_0^{y^2} \cos(y^3)\,dx\,dy = \frac{1}{3}\sin(64)$

14a $\rho = 2\sin^2\phi\cos\phi = \sin(2\phi)\sin\phi$ **b** $y^2 = 1 + 2x$

15 Let $x = au$, $y = bv$ $\frac{\partial(x,y)}{\partial(u,v)} = ab$ AREA$= \int\int_{u^2+v^2 \leq 1} ab\;du\,dv = ab\int_0^{2\pi}\int_0^1 r\,dr\,d\theta = \pi ab$