Final Examination Calculus III

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1. Approximate $\int_0^{0.5} e^{-x^3}\,dx$ with 4 decimal place accuracy. (6%)

2. Obtain the first five non-zero terms of the Maclaurin series for :

   (a) $f(x) = \dfrac{x}{2x-5}$ (3%)

   (b) $g(x) = e^x \sin x$ (3%)

3. Consider $\dfrac{x - \sin x}{x^3 \cos x}$

   (a) Expand the numerator and denominator separately using power series. (4%)

   (b) Use your result for **3a** to calculate $\lim\limits_{x\to 0} \dfrac{x-\sin x}{x^3 \cos x}$. (2%)

4. Using the Binomial series for $f(x) = \sqrt{1+x}$ to approximate $\sqrt{105}$ with $|error| < 10^{-5}$. (6%)

5. Consider the space curve $\mathbf{R}(t) = (\mathbf{t}, \mathbf{2}\cos\mathbf{t}, \mathbf{2}\sin\mathbf{t})$

   (a) Draw a rough sketch of the curve. $(0 \le t \le 2\pi)$ (2%)

   (b) Write the parametric equations of the tangent line to the curve at $\mathbf{P}(0,2,0)$. (2%)

   (c) Calculate the length of the curve from $\mathbf{P}(0,2,0)$ to $\mathbf{Q}(\pi,-2,0)$. (2%)

   (d) Find a_T , a_N , $\hat{\mathbf{T}}$ and $\widehat{\mathbf{N}}$. (4%)

   (e) Determine the curvature at $P(0,2,0)$. (2%)

6. Given the curve $x = t^2 - 4t$, $y = t^3 - 3t^2$ (at right)

   (a) Find the coordinates of all the points at which the curve is not smooth. (1%)

   (b) Explain why the curve has no vertical tangent lines. (2%)

   (c) Write, but **DO NOT EVALUATE ,** an integral for the length of the curve in the second quadrant. (Hint : Find the values of t for which the curve crosses the axes.) (2%)

   (d) Demonstrate that the curve is concave up in the second quadrant. (Hint : Find $\frac{d^2y}{dx^2}$.) (2%)

7. Calculate the area which lies inside the circle $r = 3\sin\theta$ and outside the limacon $r = 2\sqrt{2} - \sin\theta$ (depicted at right) (4%)

8. Consider $f(x,y,z) = x^2y - 2x + y - z^2 + 8 = 0 \quad \mathbf{P}_0 = (1,-1,2)$.

(a) Write the equation of the tangent plane at P_0. (3%)

(b) Find the derivative of f in the direction of $\vec{\mathbf{v}} = (2,1,-1)$ at $\mathbf{P}_0$. (2%)

(c) Determine the maximum rate of increase of f at $\mathbf{P}_0$ and the direction in which it occurs. (3%)

9. $z = f(x,y)$, $x = r\cos\theta$, $y = r\sin\theta$

(a) Find $\dfrac{\partial z}{\partial r} + \dfrac{\partial z}{\partial \theta}$ (2%)

(b) Demonstrate $\left(\dfrac{\partial z}{\partial x}\right)^2 + \left(\dfrac{\partial z}{\partial y}\right)^2 = \dfrac{1}{r^2}\left(\dfrac{\partial z}{\partial \theta}\right)^2 + \left(\dfrac{\partial z}{\partial r}\right)^2$ (3%)

10. Find and classify the critical points of $f(x,y) = x^4 + y^2 - 8x^2 - 6y + 2$ (5%)

11. Find the maximum and minimum values of $f(x,y,z) = 6x + 3y + 2z - 7$ on the ellipsoid $4x^2 + 2y^2 + z^2 = 70$. (6%)

12. In the following problem, set up but

DO NOT ATTEMPT TO EVALUATE THE INTEGRALS.

(a) (i) Roughly sketch the region bounded by the parabolic cylinder $y = 1 - x^2$ and the planes $z = 0$, $z = 3$ and $y = 0$. (1%)

(ii) Write an integral to calculate its volume. (4%)

(b) (i) Roughly sketch the region bounded above by the cone $z = \sqrt{x^2 + y^2}$ and below by the sphere $x^2 + y^2 + z^2 = z$ (1%)

(ii) Write an integral using spherical coordinates to calculate its volume. (4%)

13. Evaluate

(a) $\displaystyle\int_0^{\sqrt{2}} \int_y^{\sqrt{4-y^2}} e^{x^2+y^2}\,dx\,dy$ (5%)

(b) $\displaystyle\int_0^{16} \int_{\sqrt{x}}^{4} \cos(y^3)\,dy\,dx$ (5%)

14. Perform the following conversions (simplify as much as you can).

(a) Write in spherical coordinates: $(x^2 + y^2 + z^2)^2 = 2z(x^2 + y^2)$ (2%)

(b) Write in Cartesian coordinates: $r = \dfrac{1}{1 - 2\cos\theta}$ (2%)

15. Establish the formula $A = \pi ab$ for the area enclosed by the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ using a double integral and a change of variable. (5%)