Calculus III Final Examination

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Exhibit your work. You may assume the following:

$\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$, $\cos x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}$

$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$, $\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$

1. Establish a power series representation for $y = \arctan x$. (6%)

$$\frac{1}{1-t} = \Sigma_{k=0}^{\infty} t^k \quad |t| < 1$$

$$\frac{1}{1+t} = \Sigma_{k=0}^{\infty} (-t)^k \quad |-t| < 1 \quad i.e. \; \frac{1}{1+t} = \Sigma_{k=0}^{\infty} (-1)^k t^k \quad |t| < 1$$

$$\frac{1}{1+t^2} = \Sigma_{k=0}^{\infty} (-1)^k \, t^{2k} \quad |t^2| < 1 \iff |t| < 1$$

$$\arctan x = \int_0^x \frac{1}{1+t^2}\, dt = \Sigma_{k=0}^{\infty} (-1)^k \int_0^x t^{2k}\, dt \quad |x| < 1$$

$$= \Sigma_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{2k+1} \quad |x| < 1$$

2. Consider $f(x) = \int_0^x \sin(t^2)\, dt$

   (a) Represent $f(x)$ as a power series. (7%)

$$\sin t = \sum_{k=0}^{\infty} \frac{(-1)^k t^{2k+1}}{(2k+1)!} \quad \forall t \epsilon R$$

$$\sin(t^2) = \sum_{k=0}^{\infty} \frac{(-1)^k t^{4k+2}}{(2k+1)!} \quad \forall t \epsilon R$$

$$f(x) = \int_0^x \sin(t^2)\,dt = \sum_{k=0}^{\infty} \frac{(-1)^k \int_0^x t^{4k+2}\,dt}{(2k+1)!} \quad \forall x \epsilon R$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k x^{4k+3}}{(4k+3)\,(2k+1)!} \quad \forall x \epsilon R$$

(b) Compute $f^{(7)}(0)$. (3%)

$$f(x) = \Sigma_{k=0}^{\infty} \frac{f^{(k)}(0)x^k}{k!} = \sum_{k=0}^{\infty} \frac{(-1)^k x^{4k+3}}{(4k+3)\,(2k+1)!}$$

$7 = 4k + 3 \Rightarrow k = 1$

$$\frac{f^{(7)}(0)x^7}{7!} \equiv \frac{(-1)^1 x^{4(1)+3}}{(4(1)+3)\,(2(1)+1)!}$$

$f^{(7)}(0) = -\frac{7!}{7(3!)} = -120$

(c) Estimate $f(1)$ to 5 decimal places. (5%)

$f(1) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(4k+3)\,(2k+1)!}$; a (convergent) alternating series

The truncation error is less than the first term neglected.

$$|Error| < |a_{n+1}| = \frac{1}{(4n+7)\,(2n+3)!}$$

If $n = 1$, $\frac{1}{(4+7)\,(2+3)!} = 7.5758 \times 10^{-4} > 5 \cdot 10^{-6}$

If $n = 2$, $\frac{1}{(8+7)\,(6+3)!} = 1.8372 \times 10^{-7} < 5 \cdot 10^{-6}$

$$f(1) = \sum_{k=0}^{2} \frac{(-1)^k}{(4k+3)\,(2k+1)!} = .31028$$

3. Find the first four non-zero terms of the Maclaurin series for $y = e^x \cos x$ (5%)

$$e^x \cos x = \left(\sum_{k=0}^{\infty} \frac{x^k}{k!}\right)\left(\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}\right) = 1 + x - \tfrac{1}{3}x^3 - \tfrac{1}{6}x^4 + O\left(x^5\right)$$

4. Draw a rough sketch of the polar curve $r = 1 - 2\cos\theta$. Find <u>analytically</u> two (distinct) points at which the tangent is horizontal. (8%)

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\frac{d[(1-2\cos\theta)\sin\theta]}{d\theta}}{\frac{d[(1-2\cos\theta)\cos\theta]}{d\theta}} = \frac{\sin\theta(2\sin\theta)+(1-2\cos\theta)\cos\theta}{\cos\theta(2\sin\theta)-\sin\theta(1-2\cos\theta)}$$

$$= \frac{\cos\theta - 2(\cos^2\theta - \sin^2\theta)}{\sin\theta(4\cos\theta-1)} = \frac{-4\cos^2\theta+\cos\theta+2}{\sin\theta(4\cos\theta-1)}$$

$-4\cos^2\theta + \cos\theta + 2 = 0$ if $\cos\theta = \frac{-1\pm\sqrt{33}}{-8}$

$\frac{dy}{d\theta} = 0$ if $\theta = \arccos(-.59307)$ or $\arccos(.84307)$

In polar coordinates, two such points are $[-.68614, .84307]$ and $[-.18614, .56783]$.

(Note: $\frac{dx}{d\theta} \neq 0$ at those points.)

5. Write integrals for each of the following:

DO NOT ATTEMPT TO SOLVE THE INTEGRALS

(a) The perimeter of the hypocycloid

$x = \cos^3\theta \quad y = \sin^3\theta \quad \theta \in [0, 2\pi]$ (3%)

$s = \int_0^{2\pi} \sqrt{(-3\cos^2\theta\sin\theta)^2 + (3\sin^2\theta\cos\theta)^2}\ d\theta$

$= 3\int_0^{2\pi} |\cos\theta\sin\theta|\ d\theta$

(b) The area of one petal of the rose $r = 3\cos(3\theta)$ (3%)

$2 \cdot \left(\frac{1}{2}\int_0^{\frac{\pi}{6}} 9\cos^2(3\theta)\ d\theta\right) = \int_0^{\frac{\pi}{6}} 9\cos^2(3\theta)\ d\theta$

(c) the volume of the solid region above the XY plane which lies inside the sphere $x^2+y^2+z^2=4$ and outside the cylinder $x^2+y^2=1$ in:

(a) rectangular coordinates. (2%)

$4\left(\int_0^1 \int_{\sqrt{1-x^2}}^{\sqrt{4-x^2}} \sqrt{4-x^2-y^2}\ dydx + \int_1^2 \int_0^{\sqrt{4-x^2}} \sqrt{4-x^2-y^2}\ dydx\right)$

(b) cylindrical coordinates. (2%)

$\int_0^{2\pi}\int_1^2\int_0^{\sqrt{4-r^2}} r\ dzdrd\theta$

(c) spherical coordinates. (2%)

$\int_0^{2\pi} \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \int_{\csc\phi}^{2} \rho^2 \sin\phi \; d\rho d\phi d\theta$

6. Consider a particle moving on a circular path of radius b

$\overrightarrow{R}(t) = \overrightarrow{(b\cos\omega t\,, b\sin\omega t\,)}$ where $\omega = \dfrac{d\theta}{dt}$ is the constant angular velocity.

(a) Find the velocity vector and show that it is orthogonal to $\overrightarrow{R}(t)$. (3%)

$\overrightarrow{v}(t) = b\omega\overrightarrow{(-\sin\omega t, \cos\omega t)}$

$\overrightarrow{R}(t) \cdot \overrightarrow{v}(t) = \overrightarrow{(b\cos\omega t\,, b\sin\omega t\,)} \cdot b\omega\overrightarrow{(-\sin\omega t, \cos\omega t)} = 0$

(b) Find the speed of the particle. (2%)

$||\overrightarrow{v}(t)|| = |b\omega|\; ||\overrightarrow{(-\sin\omega t, \cos\omega t)}|| = |b\omega|$

(c) Find the magnitude of the acceleration vector. (2%)

$\overrightarrow{a}(t) = -b\omega^2\overrightarrow{(\cos\omega t\,, \sin\omega t\,)}$

$||\overrightarrow{a}(t)|| = b\omega^2$

(d) Demonstrate that the acceleration vector is always directed toward the centre of the circle. (2%)

$\overrightarrow{a}(t) = -b\omega^2\overrightarrow{(\cos\omega t\,, \sin\omega t\,)} = -\omega^2\overrightarrow{R}(t)$

The acceleration is oppositely directed to the radius vector.

(e) Compute the curvature. (3%)

$$\kappa(t) = \frac{||\overrightarrow{v}(t)\; X\; \overrightarrow{a}(t)||}{||\overrightarrow{v}(t)||^3} = \frac{b^2|\omega|^3}{b^3|\omega|^3} = \frac{1}{b}$$

7. $w = f(x, y)$ has continuous partial derivatives. Suppose that we substitute the polar coordinates r and θ.

Prove $\dfrac{1}{r}\dfrac{\partial w}{\partial \theta} = -f_x \sin\theta + f_y \cos\theta$. (3%)

$\dfrac{\partial w}{\partial \theta} = \dfrac{\partial f}{\partial x}\dfrac{\partial x}{\partial \theta} + \dfrac{\partial f}{\partial y}\dfrac{\partial y}{\partial \theta} = f_x(-r\sin\theta) + f_y(r\cos\theta)$

$\dfrac{1}{r}\dfrac{\partial w}{\partial \theta} = -f_x \sin\theta + f_y \cos\theta$ as required.

8. Find a unit vector in the direction in which $f(x,y) =$ $\cos \pi xy + xy^2$ increases most rapidly at $(\dfrac{1}{2}, \mathbf{1})$ (3%)

$\overrightarrow{\nabla} f = \overrightarrow{(-\pi y \sin \pi xy + y^2, -\pi x \sin \pi xy + 2xy)}$

$\overrightarrow{\nabla} f(\frac{1}{2}, 1) = \overrightarrow{(1-\pi, 1-\frac{\pi}{2})} \quad \hat{u} = \dfrac{\overrightarrow{\nabla} f(\frac{1}{2},1)}{||\overrightarrow{\nabla} f(\frac{1}{2},1)||} = \overrightarrow{(-.96627, -.25754)}$

9. Find the extreme value(s) of the function

$f(x,y) = xy - x^2 - y^2 - 2x - 2y + 4$ (5%)

$f_x = y - 2x - 2 = 0$; $f_y = x - 2y - 2 = 0$

Substituting the second equation into the first:

$\frac{x-2}{2} - 2x - 2 = 0 \Rightarrow x = -2 \Rightarrow y = -2$

$d(x,y) = (-2)(-2) - 1^2 > 0 \quad f_{xx} = -2$

A relative maximum (of 8) occurs at (-2,-2).

10. Find (local) maximum and minimum values of $f(x,y) = x^2 y$ on the line $x + y = 3$. <u>USE LAGRANGE MULTIPLIERS</u> (7%)

$\overrightarrow{\nabla} f = \overrightarrow{(2xy, x^2)}$; $\overrightarrow{\nabla} g = \overrightarrow{(1,1)}$

$2xy = \lambda$, $x^2 = \lambda \Rightarrow 2xy = x^2 \Rightarrow x = 0$ or $x = 2y$

CASE I: $x = 0$, $(Using\ g:)$ $y = 3$ $f = 0$

CASE II: $x = 2y$ $(Using\ g:)$ $y = 1$ $f = 4$

The critical values of f are 0 at $(0,3)$ and 4 at $(2,1)$.

11. Compute the volume of the tetrahedron in the first octant bounded by the coordinate planes and the plane $6x + 3y + 3z = 6$. (6%)

$\int_0^1 \int_0^{2-2x} 2 - 2x - y\ dydx = \int_0^1 2(1-x)y - \frac{3y^2}{2}|_0^{2-2x} dx$
$= \int_0^1 4(1-x)^2 - \frac{3(2-2x)^2}{2}\ dx = -2\int_0^1 (1-x)^2 dx = \frac{2}{3}(cubic\ units)$

12. Evaluate $\int_0^1 \int_y^1 \dfrac{\sin x}{x} dxdy$ (5%)

$\int_0^1 \int_0^x \dfrac{\sin x}{x} dydx = \int_0^1 \sin x\ dx = 1 - \cos 1\ \simeq .4597$

13. Calculate the volume of the region enclosed by the cylinder $x^2 + y^2 = 4$ and also bounded above by the paraboloid $z = x^2 + y^2$ and below by the XY plane. (5%)

$\int_0^{2\pi} \int_0^2 \int_0^{r^2} r\ dzdrd\theta = 8\pi \simeq 25.133\ (units^3)$

14. Calculate $\int\int\int_E z^2\ dx\, dy\, dz$ where E is the solid region $\dfrac{x^2}{4} + \dfrac{y^2}{9} + \dfrac{z^2}{16} \leq 1$ (8%)

Take $x = 2u, y = 3v\ and\ z = 4w$

$\dfrac{\partial(x,y,z)}{\partial(u,v,w)} = 24\ \ Integral = \int\int\int_{T(E)} 16w^2 \cdot 24\ du\, dv\, dw$

with $T(E)$ the solid region $u^2 + v^2 + 1 \leq 1$

Next transforming to spherical:

$Integral = 384\int_0^{2\pi}\left[\int_0^{\pi}\left(\int_0^1 (\rho^2\cos^2\phi)\ (\rho^2 \sin\phi)\ d\rho\right) d\phi\right]\ d\theta$

$= 384(2\pi)\frac{1}{5}\frac{2}{3} = \frac{512\pi}{5} \simeq 321.7$