Towards a notion of
Cartesian differential storage category

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Dedication

To our birthday boy, and my long-time collaborator, Robin Cockett as our collaboration enters its maturity (21 years), and he enters his dotage (60 years) . . .

Best wishes for many more productive and enjoyable years!
Preludium

2006  Differential categories—an additive monoidal category of “linear” maps, a (suitable) comonad whose coKleisli maps are “smooth”, and a differential combinator. (This gave a “categorical reconstruction” of Ehrhard & Regnier’s work)

2007  Talks by JRBC and RAGS on storage, etc (Eg my FMCS talk at Colgate)

2009  Cartesian differential categories—a left additive Cartesian category with a differential operator, and subcategories of “linear” maps. CDCs are the coalgebras of a “higher order chain rule fibration” comonad Faà [2011].
**Preludium**

**True:** The coKliesli category of a (suitable) differential (storage) category is a Cartesian differential category\[2009].

**Wished:** The linear maps of a Cartesian differential category form a differential category

**Wished:** Any Cartesian differential category may be (ff) embedded into the coKliesli category of a (suitable) differential category.

**Wished:** Any differential category may be represented as the linear maps of a (suitable) Cartesian differential category.
Preludium

With two notions of differential categories (and their ancillary notions) it’ll be convenient (today at least) to put an adjective in front of the tensor notion (‘‘⊗-differential category’’), to match that in front of the Cartesian notion. SO:

True: The coKliesli category of a (suitable) ⊗-differential (storage) category is a Cartesian differential category[2009].

Wished: The linear maps of a Cartesian differential category form a ⊗-differential category

Wished: Any Cartesian differential category may be (ff) embedded into the coKliesli category of a (suitable) ⊗-differential category.

Wished: Any ⊗-differential category may be represented as the linear maps of a (suitable) Cartesian differential category.
Summary

**Cartesian Storage Categories** given three equivalent ways:

- in terms of a system of $\mathcal{L}$-linear maps
- abstract coKleisli category
- coKleisli category of a **forceful** comonad

We can define a **Cartesian linear category** to be

- the linear maps of a Cartesian storage category
- equivalently, a Cartesian category with an exact **forceful** comonad
Summary

We define a notion of $\otimes$-representability, similar to the characterization of linear maps in terms of bilinear maps. Then TFAE:

- Cartesian storage category with $\otimes$-representability
- the coKleisli category of a $\otimes$-storage category (AKA a “Seely category”)

In this context we define a $\otimes$-linear category as the linear maps of a Cartesian storage category with $\otimes$-representability, which is equivalent to being an exact $\otimes$-storage category.
Add a **deriving transform** to a CSC to create a **Cartesian Differential Storage category**, defined by:

For a Cartesian storage category $\mathcal{X}$, TFAE:

- $\mathcal{X}$ is a CDC and Diff-linear $= \mathcal{L}$-linear
- $\mathcal{X}$ is a CDC and $\langle 1, 0 \rangle D_\mathcal{X}[\varphi]$ is $\mathcal{L}$-linear
- $\mathcal{X}$ has a deriving transformation

If linear idempotents split, this is also equivalent to being the coKleisli category of a $\otimes$-differential storage category.

More precisely: if linear idempotents split, then a Cartesian differential storage category automatically has $\otimes$-representability.
Systems of $\mathcal{L}$-linear maps

Given a Cartesian (i.e. having finite products) category $\mathbb{X}$, denote the simple slice fibration by $\mathbb{X}[\ ]$ (so $\mathbb{X}[A]$ has the same objects as $\mathbb{X}$, and morphisms $X \to Y$ are $\mathbb{X}$-morphisms $X \times A \to Y$).

$\mathbb{X}$ has a system of $\mathcal{L}$-linear maps (or “a system of linear maps”, $\mathcal{L}$ being understood) if in each simple slice $\mathbb{X}[A]$ there is a class of maps $\mathcal{L}[A] \subseteq \mathbb{X}[A]$, (the $\mathcal{L}[A]$-linear maps), satisfying:

[LS.1] Identity maps and projections are in $\mathcal{L}[A]$, and $\mathcal{L}[A]$ is closed under ordered pairs;

[LS.2] $\mathcal{L}[A]$ is closed under composition and whenever $g \in \mathcal{L}[A]$ is a retraction and $gh \in \mathcal{L}[A]$ then $h \in \mathcal{L}[A]$;

[LS.3] all substitution functors $\mathbb{X}[B] \xrightarrow{\mathbb{X}[f]} \mathbb{X}[A]$ (given by $A \xrightarrow{f} B$) preserve linear maps.

Note that $\mathcal{L}[\ ]$ is a Cartesian subfibration of $\mathbb{X}[\ ]$. 
A system of $\mathcal{L}$-linear maps is strongly classified if there is an object function $S$ and maps $X \xrightarrow{\varphi} S(X)$ such that for every $f: A \times X \rightarrow Y$ there is a unique $f^\#: A \times S(X) \rightarrow Y$ in $\mathcal{L}[A]$ (i.e. $f^\#$ is linear in its second argument) making

$$
\begin{array}{c}
A \times X \\
\downarrow 1 \times \varphi \\
A \times S(X)
\end{array} \xrightarrow{f} Y
\quad \xrightarrow{f^\#} \quad
\begin{array}{c}
A \times S(X) \\
\downarrow \varphi
\end{array}
$$

commute. The classification is said to be persistent in case whenever $f: A \times B \times X \rightarrow Y$ is linear in its second argument $B$ then $f^\#: A \times B \times S(X) \rightarrow Y$ is also linear in its second argument.

$^1$Co-classified?
**Cartesian storage categories**

**Definition**: A *Cartesian storage category* is a Cartesian category $\mathbb{X}$ with a persistently classified system of $L$-linear maps.

**Consequences** Define
\[ \epsilon = 1^\#: S(A) \to A \] as the linear lifting of the identity on $A$,
\[ \theta = \varphi^\#: A \times S(X) \to S(A \times X) \] as the linear lifting of $\varphi$,
\[ \delta = (\varphi \varphi)^\#: S(A) \to S^2(A) \] as the linear lifting of $\varphi \varphi$,
and $\mu = \epsilon_S: S^2(A) \to S(A)$. Then:

1. $S$ is a strong functor (with strength given by $\theta$).
2. $(S, \varphi, \mu)$ is a commutative monad.
3. $(S, \epsilon, \delta)$ is a comonad on the category of linear maps.
Strong abstract coKleisli categories

Definition: A strong abstract coKleisli category is a Cartesian category $\mathcal{X}$ equipped with a strong functor $S$, a strong natural transformation $\varphi : X \rightarrow S(X)$, and an unnatural transformation $\epsilon : S(X) \rightarrow X$, satisfying

1. $\epsilon_S : S^2(X) \rightarrow S(X)$ is a strong natural transformation
2. $\varphi \epsilon = 1; S(\varphi) \epsilon = 1, \epsilon \epsilon = S(\epsilon) \epsilon$
3. projections are $\epsilon$-natural

In such a category, the $\epsilon$-natural maps form a system of linear maps, classified by $(S, \varphi)$. In this case, persistence = “$S$ is a commutative monad”. But:

Fact: In a strong abstract coKleisli category, the monad $S$ is commutative, and so the classification of linear maps is persistent.

So: If $\mathcal{X}$ is Cartesian, then it is a Cartesian storage category iff it is a strong abstract coKleisli category.
Forceful comonads

We look more carefully at the comonad $S$ on the linear maps: the strength $\theta$ is not a strength on the category of linear maps, so $S$ as a comonad is not necessarily strong. We remedy this by assuming the existence of a force on the comonad $S$, viz a natural transformation $\psi: S(A \times S(X)) \to S(A \times X)$ which generates a strength in the coKleisli category making $S$ a strong monad. (There are 6 axioms on $\psi$ that do this.) A comonad with a force is called forceful.

**Proposition:** In any Cartesian storage category, the comonad $S$ on the linear maps has a force given by $\psi = S(\theta)\epsilon$ (the canonical coKleisli image of $\theta$).

**Proposition:** Given a Cartesian category with a forceful comonad, its coKleisli category is a Cartesian storage category.
Theorem: Cartesian Storage Categories

A category is a Cartesian storage category iff it is the coKleisli category of a forceful comonad iff it is a strong abstract coKleisli category.

Moreover: a category is the linear maps of a Cartesian storage category iff it is a Cartesian category with an exact forceful comonad (it’s tempting to call such categories “linear” . . . )

where “exact” means that

\[
\begin{array}{ccc}
S(S(X)) & \xrightarrow{S(\epsilon)} & S(X) \\
\downarrow{\epsilon} & & \downarrow{\epsilon} \\
S(X) & \xrightarrow{\epsilon} & X
\end{array}
\]

is a coequalizer. (A category with an exact comonad is always the subcategory of \(\epsilon\)-natural maps of its coKleisli category.)
Representing tensors

A Cartesian storage category is $\otimes$-representable\(^2\) if, in each slice $\mathbb{X}[A]$, for each $X$ and $Y$ there is an object $X \otimes Y$ and a bilinear map $\varphi \otimes : X \times Y \to X \otimes Y$ such that for every bilinear map $g : X \times Y \to Z$ in $\mathbb{X}[A]$ there is a unique linear map (in $\mathbb{X}[A]$) making the following diagram commute:

$$
\begin{array}{c}
X \times Y \xrightarrow{g} Z \\
\varphi \otimes \downarrow \\
X \otimes Y
\end{array}
$$

Note that this means in $\mathbb{X}$ we have

$$
\begin{array}{c}
A \times X \times Y \xrightarrow{g} Z \\
1 \times \varphi \otimes \downarrow \\
A \times (X \otimes Y)
\end{array}
$$

\(^2\)We are sorely tempted to call these Bilinear Cartesian Storage Categories!
Persistence

There is a corresponding notion of **unit representable**, which we shall always assume when assuming $\otimes$-representability; also we say $\otimes$-representability is **persistent** if linearity in other parameters (in $A$) is preserved.

It turns out that persistence is automatic in coKleisli categories, and so in Cartesian storage categories.

**Proposition:** If $X$ has a system of linear maps with persistent $\otimes$-representation, then $\otimes$ is a symmetric tensor product with unit on the subcategory of linear maps. Furthermore $S$ is a monoidal functor.

(The proof uses the universal lifting property given by $\otimes$-representability, in “evident ways”.)
⊗-Storage categories

A ⊗-storage category (aka “Seely” category) is a symmetric monoidal category with:

- products and
- a comonad \((S, \epsilon, \delta)\)
- which has a storage natural isomorphism \(s: S \to S\), i.e. a comonoidal transformation from \(\mathbb{X}\) as a smc wrt \(\times\) to \(\mathbb{X}\) as a smc wrt \(\otimes\).

This gives natural isos

\[
s_1: S(1) \to \top \quad \text{and} \quad s_2: S(X \times Y) \to S(X) \otimes S(Y)
\]

satisfying “obvious” coherence conditions.

(These categories are precisely what is needed to model MELL without \(\times\) but with products.)
**⊗-linear categories**

**Proposition:** The linear maps of a Cartesian storage category with ⊗-representability form a ⊗-storage category; conversely, the coKleisli category of a ⊗-storage category is a Cartesian storage category with ⊗-representability.

The second claim follows from showing that the comonad is canonically forceful; the force is given by

\[
S(A \times S(X)) \xrightarrow{s_2} S(A) \otimes S^2(X) \xrightarrow{1 \otimes \epsilon} S(A) \otimes S(X) \xrightarrow{s_2^{-1}} S(A \times X)
\]

We will say \( \mathbb{X} \) is a \( \otimes \)-linear category if it is an exact \( \otimes \)-storage category, or equivalently, the linear maps of a \( \otimes \)-representable Cartesian storage category.
To repeat:

**Proposition:** A \(\otimes\)-representable Cartesian storage category is precisely the coKleisli category of a \(\otimes\)-storage category.

**Proposition:** Any \(\otimes\)-linear category may be represented as the subcategory of linear maps of a \(\otimes\)-representable Cartesian storage category.

It now remains to add a differential operator to this setup . . .
Proposition: Suppose $\mathcal{X}$ is a Cartesian storage category. TFAE:

1. $\mathcal{X}$ is a Cartesian differential category for which $\text{Diff-linear} = \mathcal{L}\text{-linear}$
2. $\mathcal{X}$ is a Cartesian differential category for which $\eta = \langle 1, 0 \rangle D_{\times}[\varphi]$ is $\mathcal{L}\text{-linear}$
3. $\mathcal{X}$ has a deriving transformation

where in a CDC, $f$ being Diff-linear means $D_{\times}[f] = \pi_0 f$, and where a deriving transformation is an (unnatural) transformation $d_{\times}: A \times A \to S(A)$ satisfying a number (10!) of axioms.

A Cartesian differential storage category is a CSC satisfying any of the above equivalent conditions.
Deriving transformation axioms

[cd.1] \( d \times S(0) \varepsilon = 0 \) and \( d \times S(f + g) \varepsilon = d \times (S(f) + S(g)) \varepsilon \)

[cd.2] \( \langle h + k, f \rangle d \times = \langle h, f \rangle d \times + \langle k, f \rangle d \times \) and \( \langle 0, f \rangle d \times = 0 \)

[cd.3] \( d \times \varepsilon = \pi_0 \)

[cd.4] \( d \times S(\langle f, g \rangle) \varepsilon = d \times \langle S(f) \varepsilon, S(g) \varepsilon \rangle \)

[cd.5] \( d \times S(fg) \varepsilon = \langle d \times S(f) \varepsilon, \pi_1 f \rangle d \times S(g) \varepsilon \)

[cd.6] \( \langle \langle g, 0 \rangle, \langle h, k \rangle \rangle d \times S(d \times S(f) \varepsilon) \varepsilon = \langle g, k \rangle d \times S(f) \varepsilon \)

[cd.7] \( \langle \langle 0, h \rangle, \langle g, k \rangle \rangle d \times S(d \times S(f) \varepsilon) \varepsilon = \langle \langle 0, g \rangle, \langle h, k \rangle \rangle d \times S(d \times S(f) \varepsilon) \varepsilon \)

[cd.8] \( \langle 1, 0 \rangle d \times \) is \( \varepsilon \)-natural

[cd.9] \((\varepsilon \otimes 1)s_2^{-1}S(d \times) \varepsilon = (S(\varepsilon) \otimes 1)s_2^{-1}S(d \times) \varepsilon\)

[cd.10] \((\eta \otimes 1)\nabla = (\eta \otimes 1)s_2^{-1}S(d \times) \varepsilon\)

where \( \nabla \) is the codiagonal in the canonical bialgebra structure on a \( \otimes \)-storage category (see our [2006] paper for details).
Split linear idempotents

Any Cartesian storage category in which linear idempotents split in every slice and which has “codereliction” \( \eta: A \rightarrow S(A) \) (a natural transformation which is linear and splits \( \epsilon \), so \( \eta_A \epsilon_A = 1_A \)) automatically is \( \otimes \)-representable.

Hence if linear idempotents split, being a Cartesian differential storage category is equivalent to being the coKleisli category of a \( \otimes \)-differential \( \otimes \)-storage category.

And the linear maps of such a Cartesian differential storage category form a \( \otimes \)-differential \( \otimes \)-storage category.
Conclusion

So, in the “abstract” world we have the correspondance we hoped for. What remains is the embedding theorems which put “concrete” differential categories (Cartesian and $\otimes$) into that abstract world.
To be continued ...