Aspects of Cartesian Differential Categories

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Preliminaries (→ cartesian differential categories)

Left additive category: each hom set is a commutative monoid

\[ f(g+h) = fg + fh \]
\[ f0 = 0 \]

\((\text{semi})\)

A map \( f \) is left additive if also

\[(fg)h = fh + gh\]
\[oh = 0\]

Prop. The additive maps of a left additive category form an additive subcategory \( Y_+ \).

The inclusion \( Y_+ \to Y \) reflects isos.

E.g. Commutative monoids with "set" maps from a left additive, not additive, category

Left additive: because operations are defined pointwise
Not additive: because maps need not preserve monoid str.
Those that do are in \( Y_+ \).

Motivating example: Vector spaces with differentiable & additive maps
Cartesian left additive category: has products satisfied:
- $\pi_0, \pi_1, \Delta$ are additive
- $f, g$ additive $\Rightarrow f \times g$ is additive

Equivalent:
- $\pi_0, \pi_1$ additive
- $f, g$ additive $\Rightarrow \langle f, g \rangle$ is additive

Equivalent: $X$ has biproducts; $X \leftarrow X$ creates products

Equivalent: each obj $A$ has a chosen commutative monoid structure $(+; 0)$ "compatible with $X" \begin{align*} +_A &= \langle \langle x_1 + x_2 \rangle_T, \langle x_1 + x_2 \rangle_{\pi_1} \rangle \quad &0_A &= \langle 0_A, 0_A \rangle \end{align*}

Cartesian left additive functor: $\text{pres} +, 0$, products

Fact: $S$ a commutative on $\text{Cart}$ left add, $\text{cat } X \Rightarrow \pi_0$ is Cart left add, the canonical right adjoint $G \colon X \rightarrow \pi_0$ is Cart left add.

"Localisation": Simple slice category $\text{cat } X[A] = \text{the co-Kleisli cat of } A \times -$

Prelude: A "bundle" category $B(X)$
(Where $X$ is a left additive category)

Object: $(X, A)$ of $X$ $A \leftarrow X \times A$

Morphism: $\downarrow (F, f)$ $\downarrow f$
$(Y, B)$ $\leftarrow Y \times B$

Composition: $(F, f)(G, g) = (\langle f, 1 \rangle, \langle f, g \rangle)$

Identity: $(\pi_0, 1)$

$B(X)$ has additive structure $0 = (0, 0)$

$(F, f) + (G, g) = (F + G, f + g)$

This is additive in the first arg.

If the cat $X$ is Cartesian left additive, $B(X)$ has products

$1 = (1, 1)$ $\langle X, A \rangle \leftarrow (X \times Y, A \times B) \rightarrow (Y, B)$

$\text{in } X$ $\leftarrow \text{in } Y$

In fact: If $X$ is Cartesian left additive, so is $B(X)$

More: the projection $B(X) \rightarrow X$ is a Cartesian left additive functor, and is a fibration, whose fibres are additive cats.

So...
A bundle fibration is a fibration satisfying:

- \( X \) is left additive
- fibres \( p^{-1}A \) are additive (in \( \text{Obj} A \))
- \( f^*: Y_B \rightarrow Y_A \) is additive (if \( mg f: A \rightarrow B \))
- For any \( f: A \rightarrow B \) in \( X \), any \( \text{obj} X \) of \( Y_B \) ("\( X \) over \( B \)"), the domain of the cartesian lifting of \( f \) to \( X \) depends only on \( X \) and \( A \):

\[
\begin{align*}
  f^X: & \; \alpha(X,A) \\
  & \downarrow \; f \\
  A & \rightarrow \; B
\end{align*}
\]

So: If \( X \) is left additive, \( B(X) \xrightarrow{p} X \) is a bundle fibration.

If \( X \) is Cartesian left additive, so \( p \) a Cartesian left additive functor.

Suggest we have a left additive section of \( p \):

\[
\begin{align*}
  B(X) & \xrightarrow{d} X \\
  d(A) & = (d_A, A) \\
  d(f) & = (D[f], A)
\end{align*}
\]

This has some interesting consequences:

- Eg. \( d(f) d(g) = d(fg) \) becomes (consider \( 1^{st} \) component)
  \[
  \langle D[f], \pi, f \rangle D[g] = D[fg]
  \]
  \[\text{chain rule}\]

- Eg. \( d(f + g) = df + dg \Rightarrow D[f + g] = D[f] + D[g] \)

- Eg. \( d(<f, g>) = <df, dg> \Rightarrow D[f, g] = D[f, D[g] \rangle \)

- Eg. \( d(\pi) = 1 = (\pi_0, 1) \Rightarrow D[\pi] = \pi_0 \)
  \[
  \begin{align*}
  d(\pi_0) & = (\pi_0, \pi_0, \pi_0) \Rightarrow D[\pi_0] = \pi_0 \pi_0 \pi_0 \\
  \text{(and similarly)} D[\pi_1] = \pi_0 \pi_1 \pi_1
  \end{align*}
  \]

- If you recall my CTAC talk
  these are just the "derivative rules"
  I listed them...
There's even more structure lying around here:

Call a map \( f \) in \( \mathcal{X} \) linear if there is a map \( f' \) so that \( D[f] = \Pi_0 f' \)

Then:
- such a map \( f' \) is uniquely determined by \( f \) (since \( \Pi_0 \) is epi, having section \( \langle 1, 0 \rangle \)); call it \( d_0(f)(\cdot, f') \)
- \( d_0 \) is a functor; and linear maps form an additive subcategory
- if \( f \) is additive, so is \( d_0(f) \)
- projections are linear (see last slide); if \( f, g \) linear, so is \( \langle f, g \rangle \)

If we add the condition that \( d_0 \) is the identity, we can push this further; in particular \( \langle 1, 0 \rangle D[f] \) is linear (for any \( f \)) is (a variant of) [CD.6]

This can be pushed a bit further, via the Faà di Bruno categories:
\[
\begin{align*}
X & \xrightarrow{d} B(X) \xrightarrow{\partial} B^2(X) \xrightarrow{\partial} B^3(X) \xrightarrow{d} \ldots
\end{align*}
\]
where composition is given by the higher order chain rules.

\( B^4(X) \) has triplets for objects, maps

composition ("glue these circuits") in the 2nd order chain rule at the "top":
\[
G''(F'(x, z), F'(y, z), F(z)) + G'(F''(x, y, z), F(z))
\]
and \( B(X) \) composition at lower level.

---

**Introduction to Differential Categories**

- Motivating example of linear logic
  \[
  A \Rightarrow B = !A \to B
  \]
  - co-Kleisli category of cotriple \( ! \)
  - co-Kleisli maps are "smooth"

- Differential 2-calculus of Ehrhard & Regnier

- Our aim:
  - Categorically "reconstruct" the E-R differential structure
  - Set the spaces, monoidal category with comonad on it
  - Induction: the "base category" maps are "linear"

- CoKleisli maps are "smooth"
An illustration of how this works

A smooth map $f: \mathbb{R}^3 \to \mathbb{R}^2$, $f(x,y,z) = \langle x^2 + yz, z^3 - xy \rangle$

Its Jacobian

$$
\begin{pmatrix}
2x + yz & z & xy \\
-y & -z & 3z^2
\end{pmatrix}
$$

For chosen $\langle x, y, z \rangle$ this is a linear map $\mathbb{R}^3 \to \mathbb{R}^2$

So from $f: A \in \mathbb{R}^n \to \mathbb{R}^m$

we get $D[f]: A \to \text{Lin}(\mathbb{R}^n, \mathbb{R}^m)$

So in our setting we would have this:

$f: !A \to B$

$D[f]: !A \to (A \to B)$

To avoid the need for closed structure, we shall take

$D[f]: A \otimes !A \to B$

Outline

- Basic notions
  - Coalgebra modality in a (semi)additive symmetric monoidal category
  - Differential category
    - differential combinator

Examples

- Sets & relations (with "bag" functor)
- Sup lattices (with dual of free algebra functor)
- Commutative polynomials + "ordinary" derivatives (on $\text{Vec}^O$)
- Soo construction (generalises previous eg)

Extending the theory

- Storage (M. Fiore)
  - Ehrhard & Regnier (a not-necessarily-closed version of their structure)
Basic context

(semi) additive symmetric monoidal category

commutative monoid enriched
no assumption of biproducts - yet!

Coalgebra modality!

- a cotriple (comonad)
  \[ T \xleftarrow{e} !X \xrightarrow{\delta} !X \otimes !X \]
  natural coalgebra str.

- \((!X, \Delta, e)\) is a comonoid

\[ \begin{array}{ccc}
!X & \xrightarrow{\Delta} & !X \otimes !X \\
\downarrow & & \downarrow \\
!X \otimes !X & \xrightarrow{\lambda} & !X
\end{array} \]

\[ \begin{array}{ccc}
!X & \xleftarrow{e} & !X \\
\downarrow & & \downarrow \\
!X \otimes !X & \xleftarrow{\delta} & !X \otimes !X
\end{array} \] (commutes)

- \(\delta: !X \to !!X\) is a comonoid morphism

\[ \begin{array}{ccc}
!X & \xrightarrow{\delta} & !!X \\
\downarrow & & \downarrow \\
!X \otimes !X & \xrightarrow{\delta} & !!X \otimes !!X
\end{array} \] (commutes)

\[ \begin{array}{ccc}
!X & \xleftarrow{e} & !X \\
\downarrow & & \downarrow \\
!X \otimes !X & \xleftarrow{\delta} & !X \otimes !X
\end{array} \]

[we don't assume that \(\delta\), or any of these transformations are monoidal - yet]

Intuition: \(!A \to B\) is "a differentiable map \(A \to B\)"
(but we need more structure to realize this)

Storage

Given a s.m. cat with products and a comonoid \(!\) a comonoidal transformation \(s : ! \to ![X,Y] \to ![X \otimes ![Y] \to ![X] \otimes ![Y]

\[ \begin{array}{ccc}
S_k : ![1] & \to & ![T] \\
\downarrow & & \downarrow \\
S_a : ![X \otimes ![Y] & \to & ![X] \otimes ![Y]
\end{array} \]

(as a comonad)

In our setting, requiring that \(!\) be comonoidal is too strong - we'd want \(S\) to be so, but not \(e\) (the \(Id\) functor is not comonoidal)

\[ \begin{array}{ccc}
F(X \otimes Y) & \xrightarrow{\delta} & G(X \otimes Y) \\
\downarrow & & \downarrow \\
FX \otimes FY & \xrightarrow{\delta} & GX \otimes GY
\end{array} \]

\[ \begin{array}{ccc}
F(1) & \xrightarrow{s_0} & G(1) \\
\downarrow & & \downarrow \\
! & \xrightarrow{s_0} & ![T]
\end{array} \]

(no such \(s_0\) for \(G = Id\)
A (sym) monoidal cat $\mathcal{C}$ with canonical products has a storage transformation if there is a comonoidal transformation $s : ! \to ! : (X, x, 1) \to (X, \otimes, T)$ so that $s$ is comonoidal using the canonical comonoidal transformation $(X, x, 1) \to \chi_1(xY) \to \chi x1y \to !1 \to 1$

Key Fact:
For a (symm) monoidal cat with products, to have a comonoidal with (symm) storage trans in equiv. to having a (coassociative) coalgebra modality.

(11) Define $s_2 : !(X \times Y) \to !(X \times Y) \otimes !(X \times Y) \to !X \otimes !Y$
$s_2 : !(1) \to !1 \to !1 \to T$

(11) Define $s_2 : !(X \times Y) \to !(X \times Y) \otimes !(X \times Y) \to !X \otimes !Y$

Examples
- id on any cat with finite products
- $!$ in linear logic
- Dual of "algebra modality"
  - The free algebra $T(X) = \bigoplus_{k=0}^\infty X^{\otimes k}$
  - The free symmetric algebra $\text{Sym}(X) = \bigoplus_{k=0}^\infty X^{\otimes k} / S_k$
  - The "exterior algebra" $\Lambda(X) = \bigoplus_{k=0}^\infty X^{\otimes k} / A$
  - So $xy = -yx$

Differential Combinators
$D_{AB} : X(!A, B) \to X(A \otimes !A, B)$
$D_{AB}(f) = \frac{!A \to B}{A \otimes !A \to B}$ (Def)

This must satisfy:
- naturality (for combinators), additivity
- "constants have deriv = 0 " $D[e] = 0$
- product rule $(1e)(D[ef]o3) + (1e)(o3)(f \otimes D[g]) = D[(e)(f \otimes g)]$
- "Linear maps have constant deriv " $D[ef] + (1e)f$
- chain rule $D[e(3f)] = (1e)(D[e]o81f)D[f]$

There is a "circuit calculus" for all this ...
Cartesian Differential categories
- Left additive
- products
- a cartesian differential operator

\[
\begin{array}{c}
X \xrightarrow{f} Y \\
X \times X \xrightarrow{D_x(f)} Y
\end{array}
\]

Think: 1st arg is "linear"; 2nd is "smooth".

sat several axioms:

\[\begin{align*}
&\text{CD1} \quad D_x[f+g] = D_x[f] + D_x[g] \quad D_x[0] = 0 \\
&\text{CD2} \quad \langle h+k, v \rangle \ D_x[f] = \langle h, v \rangle \ D_x[f] + \langle k, v \rangle \ D_x[f] \\
&\text{CD3} \quad D_x[1] = \Pi_0 \\
&\text{CD4} \quad D_x[\langle f, g \rangle] = \langle D_x[f], D_x[g] \rangle \\
&\text{CD5} \quad D_x[fg] = \langle D_x[f], \Pi, f \rangle D_x[g] \\
&\text{CD6} \quad (\langle \Pi, f \rangle D_x)[D_x[f]] = (1 \times \Pi_0) D_x[f]
\end{align*}\]

Example: Find dimension vector spaces over \(\mathbb{R}\) (or \(\mathbb{C}\)) with infinite differentiable maps
- \(D_x\) given by Jacobian

Repeat the earlier example:
\[f : \langle x, y, z \rangle \mapsto \langle x^2+xyz, z^3-xy \rangle\]
\[D_x[f] : \langle u, v, w \rangle, \langle r, s, t \rangle \mapsto \]
\[
\begin{bmatrix}
2x+y & z & xy \\
-1 & -x & 3z^2
\end{bmatrix}
\begin{bmatrix}
u \\ v \\ w
\end{bmatrix} = \langle (2r+st)u + tv + 15w, -5u - rv + 3t^2w \rangle
\]
"Linear Maps"

In a cart diffic, f is linear if:

\[ D_x[f] = \pi_0 f \]

Properties: The linear maps form an additive subcat \( Y_{lin} \)
of a cart diffic \( X \); \( Y_{lin} \) has (bi)products; \( Y_{lin} \rightarrow X \) reflects isos & creates products.

Properties: The cokleisli category of a differential cat with biproducts is a cartesian differential cat.

Define \( D_x(f) \), for \( X \rightarrow Y \), to be:

\[
\begin{align*}
S(X \times X) \xrightarrow{\Delta} & S(X \times X) \otimes S(X \times X) \\
& \xrightarrow{S(f) \otimes S(f)} \\
& \xrightarrow{\varepsilon \circ \pi_1} \\
& S(X) \otimes S(X) \\
& \xrightarrow{D_y(f)} \\
& S(X)
\end{align*}
\]

Notation: \( D_{op} \) is the diff combinator for the diff cat.

Recall \( X[A] \) the "slice slice cat" (Cart. left adj.)

Note: each \( X[A] \) is cartesian left additive. Also:

If \( X \) is a cartesian differential cat, so is \( X[A] \) (\( \forall A \in \text{obj} X \))

- its differential \( D_x[f] = c \cdot D_{X}(f) \), the "partial derivative"

\[
\Delta = \langle \pi_0, \langle \pi_0, \pi_0, 1 \rangle \rangle \langle (0, 1) \times 1 \rangle D[f]
\]

To prove this, one could just work through the definition, but another approach suggests itself: develop a term calculus. With parameters, such a calculus naturally works "locally" - i.e., for slices.

We actually did the slice cat first, then we realized the logical approach would be "nicer"...
The Term Calculus for Cartesian differential categories

Start with the 'usual' primitive types, function symbols (types)
we can pair variables (think: our cat has products) to produce
"patterns" (like \((x,y)\)) (Assume no repeated variables)

Usual term formation rules for products, sums, substitution
(as needed for Cartesian left additive cats)

The differential term formation rule is:

\[
\Gamma, x : s \vdash t : T
\]

\[
\Gamma \vdash \frac{\partial t}{\partial x} (s) \cdot u : T
\]

Idea: \(\frac{\partial t}{\partial x}\) is the Jacobian, \((s)\) is the subst. of \(x\) for \(x\)
(giving the linear transformation) \(u\) is "application" at \(u\)
(m "dft product")

But these \(\frac{\partial t}{\partial x}\) are "partial derivatives";
I may contain other "parameters"

Example: if \(t = (ax^2 + bxyz, x^2 - xy)\) then

\[
\frac{\partial t}{\partial(x,y)} = (2ax + byz \quad bx \quad bxy \\
- y \quad - x \quad 3z^2)
\]

\[
\frac{\partial f}{\partial(x,y)} = \frac{\partial t}{\partial(x,y)} (s) \cdot u
\]

Things (usual substitution rules... plus):

(Dt.1) \(\frac{\partial}{\partial x} (t+t_z) (s) \cdot u = \frac{\partial t}{\partial x} (s) \cdot u + \frac{\partial t_z}{\partial x} (s) \cdot u\)

\(\frac{\partial}{\partial x} (s) \cdot u = 0\)

(Dt.2) \(\frac{\partial}{\partial x} (s) \cdot (u+u_z) = \frac{\partial t}{\partial x} (s) \cdot u + \frac{\partial t_z}{\partial x} (s) \cdot u\)

(Dt.3) \(\frac{\partial}{\partial x} (s \cdot t) = \frac{\partial t}{\partial x} (s) \cdot u + \frac{\partial t_z}{\partial x} (s) \cdot u\)

(Dt.4) \(\frac{\partial}{\partial x} (s) \cdot (u+u_z) = \frac{\partial t}{\partial x} (s) \cdot u + \frac{\partial t_z}{\partial x} (s) \cdot u\)

(Dt.5) \(\frac{\partial t}{\partial x} (t \cdot t_z) (s) \cdot u = \frac{\partial t}{\partial x} (t \cdot t_z) (s) \cdot u + \frac{\partial t_z}{\partial x} (t \cdot t_z) (s) \cdot u\)

(Dt.6) \(\frac{\partial t}{\partial x} (s \cdot s_z) (s) \cdot u = \frac{\partial t}{\partial x} (s \cdot s_z) (s) \cdot u + \frac{\partial t_z}{\partial x} (s \cdot s_z) (s) \cdot u\)

Facts ("Basic lemma")

- \(\frac{\partial}{\partial x} (t) (s) \cdot u = 0\) if no variable in \(x\) appears in \(t\)
- \(\frac{\partial}{\partial x} (s) \cdot (u+u_z) = \frac{\partial t}{\partial x} (s) \cdot u + \frac{\partial t_z}{\partial x} (s) \cdot u\)
- \(\frac{\partial}{\partial x} (s \cdot t) = \frac{\partial t}{\partial x} (s) \cdot u + \frac{\partial t_z}{\partial x} (s \cdot t) (s) \cdot u\)
- \(\frac{\partial}{\partial x} (t \cdot t_z) (s) \cdot (u+u_z) = \frac{\partial t}{\partial x} (t \cdot t_z) (s) \cdot u + \frac{\partial t_z}{\partial x} (t \cdot t_z) (s \cdot u)\)
- \(\frac{\partial}{\partial x} (s \cdot s_z) (s) \cdot (u+u_z) = \frac{\partial t}{\partial x} (s \cdot s_z) (s) \cdot u + \frac{\partial t_z}{\partial x} (s \cdot s_z) (s \cdot u)\)
Soundness: An interpretation \( \mathcal{M} \) of a differential theory \( T \) consists of:
- \( \{ \text{types} \} \rightarrow \text{objects} \)
- \( \{ \text{typed function symbols, equations} \} \rightarrow \text{suitably typed maps} \)

The assignment is extended as follows to all terms:

\[
\begin{align*}
[ x : A + x ] & = 1_{\mathcal{M}(A)} \\
[ x + t_1 + t_2 ] & = [ x + t_1 ] + [ x + t_2 ] \\
[ x + f(t_1 \ldots t_n) ] & = [ x + (t_1 \ldots t_n) ]_{\mathcal{M}(f)} \\
[ x + (t_1 \ldots t_n) ] & = \langle [ x + t_1 ], \ldots, [ x + t_n ] \rangle \\
\text{if } (p, p') + x & = \begin{cases} p \cdot x & p \cdot x \\
\pi_1, [ p + x ] & x \in p' \\
\end{cases} \\
\text{if } p \vdash \frac{df(x, y)}{dx} (g(a, y)) \cdot x & = \mathcal{M}(f) \\
\text{Note: } [ (a, y), x ] + f(x, y) & = [ (a, y), x ] + f(x, y) \\
& = \langle [ (a, y), x ] + f(x, y) \rangle_{\mathcal{M}(f)} \\
& = \langle [ a, y ], x + f(x, y) \rangle_{\mathcal{M}(f)} \\
\text{So } [ (a, y) + \frac{df}{dx} (g(a, y)) \cdot x ] & = \\
& = \langle \langle (a, y) + \frac{df}{dx} (g(a, y)) \cdot x, 1 \rangle \rangle_{\mathcal{M}(f)} \\
& = \langle \langle \pi_0, 0 \rangle, \langle \mathcal{M}(g), 1 \rangle \rangle_{\mathcal{M}(f)}
\end{align*}
\]

\( \text{Prop:} \) Such a translation gives a sound interpretation of terms into any cartesian differential category.
Construct the classifying cartesian differential cat of a theory "as usual":

Objects: products of primitive types
Arrows: (equiv. classes of) sequents \( x: T_1 \vdash t: T_2 \)

Composition:
\[
(x \mapsto t)(x' \mapsto t') = (x \mapsto t'[t/x'])
\]

Differential:
\[
D(x \mapsto t; x \mapsto y) = (x', x) \mapsto \frac{\partial t}{\partial x}(x) \cdot x'
\]

For example, here's the verification of the chain rule (CD.5):
\[
D[(x \mapsto t; x \mapsto s)] = D[(x \mapsto s[t/y])]
\]
\[
= (x', x) \mapsto \frac{\partial s}{\partial y}(y) \cdot (x') \cdot x'
\]
\[
= (x', x) \mapsto \frac{\partial s}{\partial y}(y) \cdot (\frac{\partial t}{\partial x}(x) \cdot x')
\]
\[
= \langle (x', x) \mapsto \frac{\partial t}{\partial x}(x) \cdot x', \pi, x \mapsto t \rangle \cdot \langle y', y \mapsto \frac{\partial s}{\partial y}(y) \cdot y' \rangle
\]
\[
= \langle D(x \mapsto t), \pi, (x \mapsto t) \rangle \cdot D(y \mapsto s)
\]
(as required)

An example of how the term calculus "cleans up" combinator terms; consider the 2nd order chain rule (seen earlier in defining composition in \( B^2(x) \))

\[
\frac{\partial^2 g f(x)}{\partial x^2} (x) \cdot x' \cdot x'' =
\]
\[
\frac{\partial^2 g}{\partial u^2} (f(x)) \cdot \left( \frac{\partial f}{\partial x}(x) \cdot x' \right) \cdot \left( \frac{\partial f}{\partial x}(x) \cdot x'' \right)
+ \frac{\partial g}{\partial u} (f(x)) \cdot \left( \frac{\partial^2 f}{\partial x^2}(x) \cdot x' \cdot x'' \right)
\]

Contrast: \( G''(F(x), F'(y), z, F(z)) + G'(F''(x, y, z), F(z)) \)

\[
\langle \langle (0, 1) \mapsto D^2[F], \langle (1, 0, 1) \mapsto D^2[F], \rangle, \langle \pi, D[f], \pi, f \rangle \rangle \ \rangle \ \rangle \ D^2[g]
\]

\[
[CD.6] \ (1 \times \pi) \ D[f]
\]
Appendix

3 slides from my FMCS 2007 talk (Colgate University, June 2007)
showing how the counored $S$ may be
drawn from the additive structure of
a left additive category.

Additional note:

During CT2007, we (B.C.S.) showed that
in a differented storage category, the
deriving transform does in fact have the universal
property of the Kähler differential $3$ so it
is a "property", not "structure", answering many
Andrew Kock's questions.

Classification & Representation

$\mathcal{X}$ is left additive cat: additive maps are weakly classified if

\[ \forall A \xrightarrow{A f} B \]

\[ \exists \phi \quad \exists f, (\text{additive}) \]

\[ S_f(x) \]

[Equivalently, $\mathcal{X} \leftrightarrow \mathcal{X}$]

[has left adjoint $S_f$]

$\mathcal{X}$ has weak binary ($\otimes$) representation if

\[ \forall X \otimes Y \xrightarrow{\text{additive}} Z \]

Also: nollary rep ($=\text{unit}$)

- strong rep ($=\text{rep for all slices}$)

- retitive rep.
The point of all this?

If $X$ is a Cartesian left additive category with retentive classification & representation, then there is a canonical comonoidal natural isomorphism

$$s : S \to S$$

$$S(A \times B) \to S(A) \otimes S(B)$$

$$S(1) \to T$$

In fact $S$ is a comonoidal monoidal (co-lax)

Then if $X$ is left additive with retentive classification & representation, then $X$ is an $S$-category

(meaning: a symmetric monoidal category with products & a comonoidal $S$ with an iso storage transformation)

---

In the following slides, we'll examine another route to such structure, allowing $S$ from the start & deriving the rest of the structure ...

Let's go back a while to revisit our alternate idea: with suitable structure on $X$, we induced $S$ on the left adjoint $S^t$ to $X \to X$ ["suitable" = retentive additive classification & tensor representation]

**Definitions:**

- An $S$-cat is a symmetric monoidal category with products, a monoidal comonoid $S$, & the $S$-isos.

- A 'prestorage' cat is a cartesian cat with retentively classified strong 'system of maps' (think: a subfiberation, closed under $\times$, of the simple fiberation given by slices $X[A]$)

- A 'storage' cat is a 'prestorage' cat until retentive tensor representation.

Then a cartesian differential category with retentive strong tensor representation has an additive strong tensor on the subcat of $S$-linear maps (given by the induced $\otimes$)

If $X$ also has retentive classification, $X_{lin}$ is a differential $S$-cat.

This is very close to what we want:

- In a 'storage' cat, the subcat of 'systemic' maps (those in the subset)
  is an $S$-cat

- The coKleisli cat of an $S$-cat is a 'pre storage' cat

- Exact $S$-cats $\leftrightarrow$ systemic maps in a 'storage' cat

- A cat, diff. cat which is a 'pre storage' cat but in which all linear idempotents split linearly is a 'storage' cat