Differential Categories II

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Introduction

- Motivating example of linear logic

$$A \Rightarrow B \;=\; !A \multimap B$$

  - coKleisli category of cotriple ! {comonad} {stable domains & coherence Spaces}

- Differential λ-calculus of Ehrhard & Regnier {Köthe spaces Finiteness spaces}

Our aim:
Categorically "reconstruct" the E-R differential Structure

Basic setting: {symmetric} monoidal category with comonad on it

Intuition: The "base category" maps are "linear"
CoKleisli maps are "smooth"

An illustration of how this works

A smooth map $f : \mathbb{R}^3 \to \mathbb{R}^2$ $\quad f\langle x,y,z\rangle = \langle x^2+xyz,\ z^3 - xy\rangle$

Its Jacobian $\begin{pmatrix} 2x+yz & xz & xy \\ -y & -x & 3z^2 \end{pmatrix}$

For chosen $\langle x,y,z\rangle$ this is a linear map $\mathbb{R}^3 \to \mathbb{R}^2$

ie from
$$\frac{f : A \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^m}{D[f] : A \longrightarrow Lin(\mathbb{R}^n, \mathbb{R}^m)}$$
we get

These are both smooth ie CoKleisli maps

So: in our setting we would have this:

$$\frac{f : \,!A \longrightarrow B}{D[f] : \,!A \longrightarrow (A \multimap B)}$$

all maps in base cat ✗

Linear Hom

To avoid the need for closed structure, we shall take

$$D[f] : A \otimes !A \longrightarrow B$$

Outline of talk

- Basic definition of differential category
  - coalgebra modality on a (semi) additive symmetric monoidal category
  - differential combinator

- Question: how can we characterize the coKleisli category of such a diff cat?
  - we shall drop the tensor structure ... coKleisli cat emphasises $\times$ over $\otimes$
  - define two secondary notions:
    - Cartesian differential categories ... coKleisli cats
    - Pre differential categories ..... $\otimes$-free diff cats
  - examine the connection between these & between prediff cats & diff cats

[Technical remark: all this is "cleanest" in the "strong" case - our focus today]

An atlas

{CoKleisli cat}

{Base cat, ×}

{Base cat, ⊗, ×}

Abstract Storage Diff. Cat $\leftrightarrows$ Storage Pre-Diff Cat $\supset$ Storage Diff Cat

$\cap$

Strong Abstract Diff Cat $\leftrightarrows$ Strong Pre-diff cat $\supset$ Strong Diff Cat

{storage transform iso}

$\cap$

{Today's Talk}

Bialgebra modality + D + ∇-rule

$\cap$

Abstract Diff Cat $\leftarrow$ Pre-diff cat $\supset$ Diff Cat

$\dashrightarrow$ (+ exactness condition)

= coalgebra modality + D

≡ storage transform + D

{CMS Talk}

Basic context

- (semi) additive symmetric monoidal category $\mathbb{X}$

commutative monoid enriched
no assumption of biproducts - yet!

Eg Sets & relations is (semi) additive but not AbGrp-enriched

- coalgebra modality !
  - a cotriple (comonad)
  - $T \xleftarrow{e} !X \xrightarrow{\Delta} !X \otimes !X$ natural coalgebra str.
  - $(!X, \Delta, e)$ is a comonoid

$$\begin{tikzcd}
!X \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & !X \otimes !X \arrow[d, "\Delta \otimes 1"] \\
!X \otimes !X \arrow[r, "1 \otimes \Delta"'] & !X \otimes !X \otimes !X
\end{tikzcd}
\qquad
\begin{tikzcd}
 & !X \arrow[dl, "1"'] \arrow[d, "\Delta"] \arrow[dr, "1"] & \\
!X & !X \otimes !X \arrow[l, "1 \otimes e"] \arrow[r, "e \otimes 1"'] & !X
\end{tikzcd}$$

commute

  - $\delta : !X \to !!X$ is a comonoid morphism

$$\begin{tikzcd}
!X \arrow[rr, "\delta"] \arrow[dr, "e"'] & & !!X \arrow[dl, "e"] \\
 & T &
\end{tikzcd}
\qquad
\begin{tikzcd}
!X \arrow[r, "\delta"] \arrow[d, "\Delta"'] & !!X \arrow[d, "\Delta"] \\
!X \otimes !X \arrow[r, "\delta \otimes \delta"'] & !!X \otimes !!X
\end{tikzcd}$$

commute

[we don't assume that δ, or any of these transformations are monoidal - yet]

Intuition: $!A \to B$ is "a differentiable map $A \to B$"

(but we need more structure to realize this)

Storage

Given a s.m.cat with products and a comonad $!$
a comonoidal transformation $s : ! \to !$
from $(\mathcal{X}, \times, 1)$ to $(\mathcal{X}, \otimes, \top)$ amounts to

$$s_0 : !(1) \to \top \quad \text{and} \quad s_2 : !(X \times Y) \to !X \otimes !Y$$

st

$$\begin{tikzcd}
!((X\times Y)\times Z) \arrow[r, "s_2"] \arrow[d, "!(a_\times)"'] & !(X\times Y)\otimes !Z \arrow[r, "s_2\otimes 1"] & (!X\otimes !Y)\otimes !Z \arrow[d, "a_\otimes"] \\
!(X\times(Y\times Z)) \arrow[r, "s_2"'] & !X\otimes !(Y\times Z) \arrow[r, "1\otimes s_2"'] & !X\otimes(!Y\otimes !Z)
\end{tikzcd}$$

$$\begin{tikzcd}
!(1\times X) \arrow[r, "s_2"] \arrow[d, "!\pi_1"'] & !(1)\otimes !X \arrow[d, "s_0\otimes 1"] \\
!X & \top\otimes !X \arrow[l, "u_\otimes"]
\end{tikzcd}
\qquad
\begin{tikzcd}
!(X\times 1) \arrow[r, "s_2"] \arrow[d, "!\pi_1"'] & !X\otimes !(1) \arrow[d, "1\otimes s_0"] \\
!X & !X\otimes \top \arrow[l, "u_\otimes"]
\end{tikzcd}$$

(+ diagram for symmetry if appropriate)

In our setting, requiring that $!$ be comonoidal (as a comonad) is too strong – we'd want δ to be so, but not ϵ (The Id functor is not comonoidal)

$$\begin{tikzcd}
F(X\times Y) \arrow[r, "\alpha"] \arrow[d, "s^F"'] & G(X\times Y) \arrow[d, "s^G"] \\
FX\otimes FY \arrow[r, "\alpha\otimes\alpha"'] & GX\otimes GY
\end{tikzcd}
\qquad
\begin{tikzcd}
F(1) \arrow[rr, "\alpha"] \arrow[dr, "s^F"'] & & G(1) \arrow[dl, "s^G"] \\
& \top &
\end{tikzcd}$$

no such s^G for $G = Id$

A (s)m cat $\mathbb{X}$ with comonad $!$, products has a <u>storage transformation</u> if there is a comonoidal transformation

$$s : ! \to ! : (\mathbb{X}, \times, 1) \to (\mathbb{X}, \otimes, \top)$$

so that δ is comonoidal

using the canonical comonoidal trans $(\mathbb{X}, \times, 1)$
ie $!(X \times Y) \to !X \times !Y$
$!(1) \to 1$

Key Fact:

For a (symm) monoidal cat with products:
to have a comonad with (symm) storage trans. is equiv. to having a (cocommutative) coalgebra modality.

($\Downarrow$) Define $\Delta : !X \xrightarrow{!(\Delta_X)} !(X \times X) \xrightarrow{s_2} !X \otimes !X$

$e : !X \xrightarrow{!(\pi)} !(1) \xrightarrow{s_0} \top$

($\Uparrow$) Define $s_2 : !(X \times Y) \xrightarrow{\Delta} !(X \times Y) \otimes !(X \times Y) \xrightarrow{!\pi_0 \otimes !\pi_1} !X \otimes !Y$

$s_0 : !(1) \xrightarrow{e} \top$

This works!

Examples

- id on any cat with finite products
- $!$ in linear logic
- Dual of "algebra modality"
  - The free algebra $T(X) = \bigoplus_{n=0}^{\infty} X^{\otimes n}$
  - The free symmetric algebra $\mathrm{Sym}(X) = \bigoplus_0^{\infty} X^{\otimes n}/S_n$
  - The "exterior algebra" $\Lambda(X) = \bigoplus_0^{\infty} X^{\otimes n}/A$

    so $xy = -yx$

Differential Combinators

$$D_{AB} : \mathbb{X}(!A, B) \longrightarrow \mathbb{X}(A \otimes !A, B)$$

$$\frac{!A \xrightarrow{f} B}{A \otimes !A \xrightarrow[D[f]]{} B}$$

... Think $!A \to (A \multimap B)$

This must satisfy:

- naturality (for combinators), additivity
- "constants have deriv = 0" $D[e] = 0$
- product rule $(1\otimes\Delta)(D[f]\otimes g) + (1\otimes\Delta)(c\otimes 1)(f\otimes D[g]) = D[\Delta(f\otimes g)]$
- "Linear maps have constant deriv" $D[\epsilon f] = (1\otimes e)f$
- chain rule $D[\delta !f g] = (1\otimes\Delta)(D[f]\otimes\delta !f)D[g]$

There is a "circuit calculus" for all this ...

Preliminaries (→ cartesian differential categories)

Left (semi) additive category : each hom set is a commutative monoid

$$f(g+h) = fg + fh$$
$$f0 = 0$$

{ diagrammatic order of composition }

A map h is (semi) additive if also

$$(f+g)h = fh + gh$$
$$0h = 0$$

Prop The additive maps of a left additive category $\mathbb{X}$ form an additive subcategory $\mathbb{X}_+$.
The inclusion $\mathbb{X}_+ \hookrightarrow \mathbb{X}$ reflects isos.

Eg Commutative monoids with "set" maps { no preservation properties }
form a left additive, not additive, category

Left additive: because operations are def^d pointwise
Not additive: because maps need not preserve monoid str.
Those that do are in $\mathbb{X}_+$.

Cartesian Differential categories

- Left additive
- products
- a cartesian differential operator

$$\frac{X \xrightarrow{f} Y}{X \times X \xrightarrow[D_\times[f]]{} Y}$$

Think: 1^{st} arg^t is "linear"; 2^{nd} is "smooth"

sat several axioms:

[CD1] $D_\times[f+g] = D_\times[f] + D_\times[g]$; $D_\times[0] = 0$

[CD2] $\langle h+k, v\rangle D_\times[f] = \langle h,v\rangle D_\times[f] + \langle k,v\rangle D_\times[f]$
$\langle 0,v\rangle D_\times[f] = 0$

[CD3] $D_\times[1] = \pi_0$; $D_\times[\pi_i] = \pi_0 \pi_i$ $(i=0,1)$

[CD4] $D_\times[\langle f,g\rangle] = \langle D_\times[f], D_\times[g]\rangle$

[CD5] $D_\times[fg] = \langle D_\times[f], \pi_1 f\rangle D_\times[g]$

<u>Example</u>: Fin dim vector spaces over $\mathbb{R}$ (or $\mathbb{C}$ ...) with ∞'ly differentiable maps

- $D_\times$ given by Jacobian

"just like" the diff cat eg

"Linear maps"

In a cart diff cat, f is linear if

$$D_\times[f] = \pi_0 f$$

Prop: The linear maps form an additive subcat $\mathbb{X}_{lin}$ of a cart diff cat $\mathbb{X}$; $\mathbb{X}_{lin}$ has (bi)products; $\mathbb{X}_{lin} \hookrightarrow \mathbb{X}$ reflects isos & creates products

Prop: The coKleisli category of a differential cat with biproducts is a cartesian differential cat

define $D_\times[f]$, for $X \xrightarrow{f} Y$, to be:

$$\begin{tikzcd}
S(X\times X) \arrow[r, "\Delta"] \arrow[rr, bend right, "{S_2 \atop \text{(The canonical storage trans}^{=}\text{)}}"'] & S(X\times X)\otimes S(X\times X) \arrow[r, "S\pi_0\otimes S\pi_1"] & S(X)\otimes S(X) \arrow[d, "\epsilon\otimes 1"] \\
 & & X\otimes S(X) \arrow[d, "D_\otimes[f]"] \\
 & & S(X)
\end{tikzcd}$$

Notation: $D_\otimes$ is the diff combinator for the diff cat
($D_\times$ is the cart diff op)
S is the comonad
(called ! in linear logic)

MORE: The coKleisli cat of a diff cat also satisfies

- $D_\times[\epsilon] = \pi_0 \epsilon$
- $D_\times[S(f)] = \pi_0 S(f)$ (for any f)

STRONG Differential categories

- Diff cats with storage transformations s_0, s_2 iso

(without further coherence - this is weaker than what we called "storage diff. cats")

In this context we can define some maps in the coKleisli cat:

$\varphi : A \to S(A)$ (ie $id : S(A) \to S(A)$ in $\mathbb{X}$)

$\eta : A \to S(A) := \epsilon(1 \otimes \iota) d_\otimes$

ie $S(A) \xrightarrow{\epsilon j u} A \otimes T \xrightarrow{1 \otimes \iota} A \otimes S(A) \xrightarrow[D_\otimes[1]]{d_\otimes} S(A)$ in $\mathbb{X}$

$d_\times : A \times A \to S(A) := D_\times[\varphi]$

ie $S(A \times A) \xrightarrow{s_2} S(A) \otimes S(A) \xrightarrow{\epsilon \otimes 1} A \otimes S(A) \xrightarrow{d_\otimes} S(A)$ in $\mathbb{X}$

PROP: The coKleisli cat of a strong diff cat satisfies:

- η is "ϵ-natural"
  [ie $\epsilon\eta = S(\eta)\epsilon$]

NB In the coKleisli cat, ϵ is NOT a nat. transformation: f is "ϵ-natural" if the ϵ-naturality square with f commutes

- $\eta = \langle 1, 0 \rangle d_\times$
- $\eta\epsilon = 1$
- "ϵ-natural" $\equiv$ "linear"
- $S(d_\times)\epsilon = s_2(\epsilon\eta \otimes 1)\nabla$
- $d_\otimes = (\eta \otimes 1) s_2^{-1} S(d_\times)\epsilon$ ...

So we can recapture $d_\otimes$ from $d_\times$ aswell as vice versa

These are properties we shall want to hold in our characterization of coKleisli cats of (strong) diff cats.

Disclaimer: The "not strong" case is more "fiddley"!

Abstract coKleisli category

{ Inspired by Carsten Führmann's charact^{n} of abstract Kleisli cats }

- cat $\mathbb{Y}$, functor $S : \mathbb{Y} \to \mathbb{Y}$
- nat. transf^{n}: $\varphi : A \to S(A)$
- <u>un</u>natural transf^{n}: $\epsilon : S(A) \to A$
  - $\epsilon\epsilon = S(\epsilon)\epsilon$ { ϵ is "ϵ-natural" }
  - $\varphi\epsilon = 1_A \qquad S(\varphi)\epsilon = 1_{S(A)}$
  - $\epsilon_{S(A)} : S(S(A)) \longrightarrow S(A)$ is natural in A

This makes $(S, \varphi, \epsilon_{S-})$ a monad/triple on $\mathbb{Y}$

Let $\mathbb{Y}_\epsilon$ be the subcat of ϵ-natural maps

Then $(S, \epsilon, S(\varphi))$ is a comonad on $\mathbb{Y}_\epsilon$

and $\mathbb{Y}$ is its coKleisli cat $(\mathbb{Y}_\epsilon)_S$

Q: Given a comonad S, what's the connection between $\mathbb{X}$ and $(\mathbb{X}_S)_\epsilon$?

(for "any" $\mathbb{X}$ with a comonad S)

Technical remark

If $\mathcal{Y}$ is an abstract coKleisli cat

① $\quad A \xrightarrow{\varphi} S(A) \underset{S(\varphi)}{\overset{\varphi}{\rightrightarrows}} S(S(A))$ is a (split) equalizer (for all A)
(by ϵ)

② $\quad S(S(A)) \underset{S(\epsilon)}{\overset{\epsilon}{\rightrightarrows}} S(A) \xrightarrow{\epsilon} A$ is a (split) coequalizer (")
(by φ)

The 2nd diagram is in $\mathcal{Y}_\epsilon$; it characterizes comonads from abstract coKleisli cats, in this sense:

<u>FACT</u>: The canonical functor (when $\mathcal{X}$ carries a comonad S)

$$\begin{tikzcd} \mathcal{X} \arrow[r] \arrow[dr, hook] & (\mathcal{X}_S)_\epsilon \\ & \mathcal{X}_S \arrow[u] \end{tikzcd}$$

is an iso iff ② is a coequalizer (for all A)

(Call such S an exact comonad)

In the "strong" context, having η forces exactness

since then η makes ② a split coequalizer {so absolute}

$$\left\{ \begin{array}{l} \text{since } \epsilon\epsilon = S(\epsilon)\epsilon \\ \qquad \eta\epsilon = 1 \\ \qquad \eta S(\epsilon) = \epsilon\eta \end{array} \right\}$$

(Without "strength", $\mathcal{X} \to (\mathcal{X}_S)_\epsilon$ need not even preserve what structure $\mathcal{X}$ has, so things become "fiddley", as we've said before)

$\mathbb{Y}$ is an abstract additive coKleisli category if in addition

- left additive ie "semi-additive"
- ϵ, $S(f)$ (for all f) are additive

[This guarantees $\mathbb{Y}_\epsilon$ is infact additive

Furthermore, the coKleisli cat of a comonad on an additive cat satisfies these, so this characterizes coKleisli cats for comonads on additive cats]

$\mathbb{Y}$ is a strong abstract differential category if in addition:

- cartesian diff. cat st π_i, Δ are ϵ-natural,
- $D_\times[\epsilon] = \pi_0 \epsilon$
- $D_\times[S(f)] = \pi_0 S(f)$ (for all f)
- $\eta := \langle 1, 0 \rangle D_\times[\varphi]$ is ϵ-natural

[These guarantee all the properties of the earlier prop. which hold of coKleisli cats of strong diff cats

– incl. "ϵ-natural" $\equiv$ "linear"]

Technical note (⤳ "predifferential categories")

In this context (somewhat less suffices) we can define a "cartesian deriving transformation"

$$d_\times : A \times A \longrightarrow S(A)$$

satisfying "the usual axioms", so that this structure is equivalent to having a cartesian differential operator $D_\times$

An impressionist's view of the axioms:

[cd1] $d_\times S(f+g)\epsilon = d_\times S(f)\epsilon + d_\times S(g)\epsilon \quad ; \quad d_\times S(0)\epsilon = 0$

[cd2] $\langle h+k, v\rangle d_\times = \langle h, v\rangle d_\times + \langle k, v\rangle d_\times \quad ; \quad \langle 0, v\rangle d_\times = 0$

[cd3] $d_\times \epsilon = \pi_0$

[cd4] $d_\times S(\langle f, g\rangle)\epsilon = d_\times \langle S(f)\epsilon, S(g)\epsilon\rangle$

[cd5] $\langle d_\times S(f)\epsilon, \pi_1 f\rangle d_\times S(g)\epsilon = d_\times S(fg)\epsilon$

STRONG Prediferential Category

$\mathbb{X}$: additive with biproducts

comonad (S, ϵ, δ)

a "predifferential operator" $d'_\times : S(A \times A) \to S(A)$

Think: this is a coKleisli map $A \times A \to S(A)$ - ie $d_\times$ in $\mathbb{X}_S$

$\eta : A \to SA$ .... we're in the "strong" setting

st

[pd 1] $\delta S(\langle h+k, v\rangle) d'_\times = \delta\left[S(\langle h, v\rangle) + S(\langle k, v\rangle)\right] d'_\times$

[pd 2] $d'_\times \epsilon = \epsilon \pi_0$

[pd 3] $d'_\times \delta S(f) g = \delta S(\langle d'_\times f, S(\pi_1) f\rangle) d'_\times g$

[spd 1] $\eta \epsilon = 1$

[spd 2] $S(\langle 1, 0\rangle) d'_\times = \epsilon \eta$

These are "coKleisli" translations of the axioms for $d_\times, \eta$

The point of this:

- The coKleisli cat $\mathbb{X}_S$ of a $\underset{\lambda}{\text{strong}}$ prediff. cat $\mathbb{X}$ is a $\underset{\lambda}{\text{strong}}$ abstract diff cat st $(\mathbb{X}_S)_\epsilon = \mathbb{X}$

must assume some exactness in the "not-strong" case

- If $\mathbb{Y}$ is a $\underset{\lambda}{\text{strong}}$ abstract diff cat, then $\mathbb{Y}_\epsilon$ is a $\underset{\lambda}{\text{strong}}$ prediff cat [and of course $(\mathbb{Y}_\epsilon)_S = \mathbb{Y}$]

$$d_X' = S(d_X)\epsilon$$

So we've characterized abstract diff cats as coKleisli cats on prediff cats

- What about diff cats?
- What happened to $\otimes$?

- Any $^{\text{strong}}$ differential cat "is" (ie induces) a $^{\text{strong}}$ pre differential cat

via:

$$d'_\times := S(A\times A) \xrightarrow{s_2} S(A)\otimes S(A) \xrightarrow{\epsilon\otimes 1} A\otimes S(A) \xrightarrow{d_\otimes} S(A)$$

diagram chase!!

Whence:

Prop: $\mathbb{X}$ is a $^{\text{strong}}$ differential cat iff

- $\mathbb{X}$ carries a $^{\text{strong}}$ coalgebra modality
- $\mathbb{X}$ is a $^{\text{strong}}$ predifferential category st

$$d'_\times = s_2\,;(\epsilon\eta\otimes 1)\,;\nabla \quad \ldots$$

Recall this property from previous slide.

$d_\otimes = (\eta\otimes 1)\nabla$

Look again at our "atlas":

$$\begin{tikzcd}
\mathbb{X}_s & \mathbb{X} \arrow[l] \arrow[r, bend left, "\text{+ coalg. modality} \atop \text{+ equations}"] & {} \\
\text{Strong Abstract Diff cat} \arrow[r, shift right] & \text{Strong Pre Diff Cat} \arrow[l, shift right] \arrow[r, phantom, "\supset"] & \text{Strong Diff Cat} \\
\mathbb{Y} \arrow[r] & \mathbb{Y}_\epsilon &
\end{tikzcd}$$

Comment about "the fiddley bits":

- stepping "down" (dropping "strong") just requires care (& some exactness assumptions at times)

- stepping "up" ("storage" – ie the linear logic!) only affects the "abstract" column: how to describe storage "abstractly" can be done by examining the interaction of $\otimes$ and $\times$ in coKleisli cats [details are in the paper!] "in preparation"

Future work?

- Examples would be nice ...
- Higher order version of this setting also ...
- Connect to other notions of differentiability ...

— Some of this is in progress, some still just "ambition" ...