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Introduction to linear bicategories

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Linear bicategories are a generalization of bicategories, in which the one horizontal composition is replaced by two (linked) horizontal compositions. These compositions provide a semantic model for the tensor and par of linear logic: in particular, as composition is fundamentally noncommutative, they provide a suggestive source of models for noncommutative linear logic.

In a linear bicategory, the logical notion of complementation becomes a natural linear notion of adjunction. Just as ordinary adjoints are related to (Kan) extensions, these linear adjoints are related to the appropriate notion of linear extension.

There is also a stronger notion of complementation, which arises, for example, in cyclic linear logic. This sort of complementation is modelled by cyclic adjoints. This leads to the notion of a $*$-linear bicategory and the coherence conditions which it must satisfy. Cyclic adjoints also give rise to linear monads: these are, essentially, the appropriate generalization (to the linear setting) of Frobenius algebras and the ambialgbras of Topological Quantum Field Theory.

A number of examples of linear bicategories arising from different sources are described, and a number of constructions which result in linear bicategories are indicated.

Contents

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0. Introduction

One of the most significant aspects of linear logic (Girard 1987) is that it offers a logical context with an involutive negation and a rich proof-theoretic structure. This contrasts with classical logic, where the equivalence relation induced by cut elimination steps "collapses" proof-theoretic structure, in the sense of making all derivations of a given sequent equivalent. The theory we begin to develop in this paper sheds additional light on the proof-theoretic content of negation in a linear setting, particularly in the noncommutative context.

To understand a phenomenon one must often start by considering its absence. Thus, to understand negation in linear logic it is instructive to consider underlying settings which do not have negation nor, indeed, any of the more elaborate structures of that logic.

The understanding of the proof-theoretic structure of a logic demands more than simply describing its sequent calculus. One needs a natural means of representing the proofs and an underlying semantics which captures the natural notion of proof equivalence. This latter may best be given in terms of a categorical doctrine. As was pointed out in (Seely 1989) the categorical semantics of linear logic essentially predated that logic in the doctrine of $*$-autonomous categories (Barr 1979). Thus the introduction of linear logic reinvigorated interest in the categorical proof theory of monoidal settings as a means to understand linear structure.

But this neglected important structure, namely the fact that there were two linked tensors present. Just as the study of semilattices fails to clarify an important aspect of Boolean negation, so a setting based on a single tensor fails to allow the expression of certain important structural aspects of linear negation. What was missing was the structural analogue of a distributive lattice, and of the transformation therein of negation into the property of being a complement. It was necessary, therefore, to understand the relationship between the tensor and the par†. The basic notion which has emerged from

† We use $\oplus$ here in preference to Girard's original $⅋$; it is still pronounced "par".

this analysis is that of a *linearly distributive* category (Cockett–Seely 1992) in which two monoidal structures are linked by "linear distributivities". These maps, originally called "weak distributivities"‡, behave simultaneously as tensorial strengths and costrengths. This setting is, in fact, the categorical semantics for the proof theory of the pure cut rule.

In this context, we argue that linear negation turns into the structural analogue of a complement and is, in fact, a natural notion of adjunction for these settings. However, this is not the only interpretation of negation. For example, intuitionistic negation is derived solely from the monoidal closed structure, using the "implication" (or "internal hom" for the tensor). This leads one to the Full Intuitionistic Linear Logic (FILL) of (Hyland–de Paiva 1993). However, notice that this "intuitionistic" negation expresses a property which is independent of the par. In (Cockett–Seely 1997a) it is pointed out that one way to address this lack of interaction is by altering the interpretation of implication. Given a sufficiently strong interaction between the implication and the par, as in Grishin's Implicative Linear Logic (GILL) and Lambek's Bilinear Logic (BILL), which share the doctrine of (not necessarily symmetric) $*$-autonomous categories, the negation introduced through implication once again becomes the above notion of complementation.

Monoidal categories are well-known to be special cases of bicategories; we generalize linearly distributive categories to be special cases of linear bicategories—bicategories with two 1-cell compositions linked by linear distributivities. The first motivation for making this further abstraction came from the bicategorical approach to $*$-autonomy and the Chu construction, presented by the second author at the Vancouver Category Theory Summer Meeting in 1997 (Koslowski 1998).

There are two significant ways in which linear bicategories help us understand the structure of noncommutative multiplicative linear logic. First, since composition is a naturally noncommutative operation, they provide a natural semantics (perhaps the first such semantics) for noncommutative linear logics. More importantly, they allow a very efficient description of negation—or rather complementation—in terms of adjunctions. While this is not the standard notion of adjunction, it is certainly the appropriate notion for linear bicategories and hence we call it a *linear* adjunction.

Furthermore, there is a stronger notion of complementation which corresponds to the negation of cyclic linear logic. In order to model this we introduce cyclic (linear) adjunctions, and we discuss the coherence conditions necessary for such a cyclic notion of negation, arriving at the structure of "$*$-linear bicategories".

We end the paper by illustrating how the analogy between standard notions and their linear counterparts may be extended, by defining the notion of a linear monad which is induced by a cyclic adjoint; this is the natural generalization to the linear setting of Frobenius algebras and, from Topological Quantum Field Theory, ambialgbras (see (Quinn 1995) for a survey).

In subsequent work we intend to develop further the constructions associated with linear bicategories. A whole range of suggestive possibilities opens up when one considers

‡ The same term was also used by (Hyland–de Paiva 1991), who independently—and at the same time—realized their fundamental role.

the ordinary bicategorical notions which could be linearized. For example, the module construction which allows one to build a bicategory from the monads of another bicategory can clearly be generalized to provide a linear bicategory from the linear monads of another linear bicategory. It is of interest to note that applying such a linear construction to a "degenerate" (in the sense that $\otimes = \oplus$) structure, such as Frobenius algebras, will result in a non-degenerate linear bicategory, where $\otimes$ and $\oplus$ are different. We also plan to further develop the Chu construction in this setting, which allows us to create new closed linear bicategories and $*$-linear bicategories from closed bicategories.

Throughout the paper we will frequently use the language of planar *circuit diagrams* as introduced in (Blute *et al.* 1996; Cockett–Seely 1997a; Cockett–Seely 1998). In order to deal with the subtleties concerning the distinction between FILL and GILL, the diagrams dealing with the monoidal closed structure are slightly more complicated than would be necessary for the purely closed linear setting—this is discussed in (Cockett–Seely 1997a). The reader should be advised that our diagrams are to be read from the top down, in contrast to the string diagrams proposed by Joyal and Street and employed by the second author in (Koslowski 1997; Koslowski 1998).

This paper is dedicated to Prof. Joachim (Jim) Lambek, as part of the celebration of his 75th birthday. Jim Lambek has been a central figure in categorical proof theory from its beginning as an identifiable field of study. His 1958 paper (Lambek 1958) on the syntactic calculus, while primarily dealing with an application to linguistics, laid out the essential basics for a categorical analysis of monoidal logics, such as linear logic. His 1966 text (Lambek 1966) on rings and modules may be viewed as a study of the canonical example of a closed bicategory. His work on deductive systems, Cartesian closed categories and type theory (Lambek 1968-72; Lambek–Scott 1986) was of course a main inspiration for most of the subsequent work on categorical proof theory done in the past three decades. More recently, his return to the study of bilinear logic (or noncommutative multiplicative linear logic) (Lambek 1993) places his work in the center of ongoing research on applications of linear logic. Currently (Lambek 1995-99) Jim has been studying what in the terms of the present paper may be described as the compact ($\otimes = \oplus$) case of closed linear bicategories, with particular applications to linguistics; (Lambek 1999) drops the compactness requirement to consider "Grishin po-categories", the posetal version of linear bicategories. On a personal note, the third author's first contact with the subject was during an introductory undergraduate course on logic in the late 1960's, when Jim used several lectures to describe his (then recent) work on Cartesian closed categories as a deductive system; the path from those undergraduate lectures to this paper is direct and continuous. We all wish him the very best, and hope to derive further inspiration from his research for many years to come.

1. A logic of generalized relations

1.1. *The logic*

An obvious modification of the basic two-sided linear sequent calculus (the tensor-par fragment of linear logic) is to interpret the formulas as generalized relations having a domain and codomain. Under this interpretation the juxtaposition of formulas in the logic will be constrained by the requirement that the codomain and domain must match. This in turn means that such logics will be noncommutative in character, since interchanging formulas will in general make no sense.

Explicitly, each formula A of this logic of generalized relations has an associated pair of types $A\colon X_0 \longrightarrow X_1$, called, respectively, its domain and codomain. To emphasize this structure, we shall use the word "relation" in preference to "formula" henceforth. Sequents are then built from typed lists of relations.

Such a list $\Gamma = [A_1, \ldots, A_n]\colon X_0 \longrightarrow X_n$ must satisfy the requirement that the codomain of A_i equals the domain A_{i+1}, for $i = 1, \ldots, n-1$. The domain of Γ must be the domain of the first relation $A_1\colon X_0 \longrightarrow X_1$ and similarly the codomain must be the codomain of the last relation $A_n\colon X_{n-1} \longrightarrow X_n$. In short, these are just directed paths. If Γ is empty then the only requirement is that the domain and codomain be the same and then the typed lists $[\,]\colon X_0 \longrightarrow X_0$ and $[\,]\colon X_1 \longrightarrow X_1$ are regarded as being distinct.

The sequents of the logic are pairs of like typed lists of relations $\Gamma \vdash \Delta\colon X_0 \longrightarrow X_n$ which are built using the rules in Table 1. Each side of a sequent is to be interpreted as a relation: the fact that a sequent is derivable means there is a proof of the second relation from the first and for this to make sense the relations Γ and Δ must be relations of the same type. Where necessary, we have indicated the typing *via* superscripts; otherwise it may be inferred from the context.

It should also be mentioned that the sequent rules in Table 1 use the categorically inspired "bidirectional" rules. These may be replaced by the more traditional "unidirectional" (left and right) introduction rules (see (Cockett–Seely 1997)) in order to secure the cut elimination property, which is clearly not satisfied by the above system. The advantage of the system above is that it does express the intended meaning of the connectives in a direct fashion. Notice also that the usual cut rule splits into four cases since, in the noncommutative situation, no change in the order of the premises and conclusions is allowed; this corresponds to the "planarity" constraint on the circuit diagrams (discussed in Appendix A).

1.2. *Categorical semantics*

From a categorical proof theoretic perspective it would be tempting to interpret this logic as a bicategory with juxtaposition of relations being 1-cell composition and proofs being 2-cells. While this is a valid interpretation, it fails to capture a rather important distinction: in a two-sided sequent calculus the juxtaposition on the left of the turnstile is traditionally provided by a *different* connective from that used to represent the juxtaposition on the right of the turnstile. Thus, for example, in ordinary propositional logic the former is interpreted as "and" while the latter is interpreted as "or". Similarly, in linear logic the former is the "tensor" (here represented by $\otimes$) while the latter is the "par" (here represented by $\oplus$)

Consider how we may interpret this logic as ordinary relations between sets: the jux-

Table 1. *Sequent rules for a logic of generalized relations*

$$\frac{}{A^{X \longrightarrow Y} \vdash A^{X \longrightarrow Y}}\ (id)$$

$$\frac{\Gamma, A^{X \longrightarrow Y}, B^{Y \longrightarrow Z}, \Gamma' \vdash \Delta}{\Gamma, A \otimes B^{X \longrightarrow Z}, \Gamma' \vdash \Delta}\ (\otimes) \qquad \frac{\Gamma \vdash \Delta, A^{X \longrightarrow Y}, B^{Y \longrightarrow Z}, \Delta'}{\Gamma \vdash \Delta, A \oplus B^{X \longrightarrow Z}, \Delta'}\ (\oplus)$$

$$\frac{\Gamma, \Gamma' \vdash \Delta}{\Gamma, \top^{X \longrightarrow X}, \Gamma' \vdash \Delta}\ (\top) \qquad \frac{\Gamma \vdash \Delta, \Delta'}{\Gamma \vdash \Delta, \bot^{X \longrightarrow X}, \Delta'}\ (\bot)$$

In these rules, the double horizontal line indicates that the inference may go either direction (top to bottom or bottom to top), *i.e.* these rules are "bijective".

$$\frac{\Gamma \vdash \Delta_1, A^{X \longrightarrow Y}, \Delta_2 \quad A^{X \longrightarrow Y} \vdash \Delta}{\Gamma \vdash \Delta_1, \Delta, \Delta_2}\ (cut_1) \qquad \frac{\Gamma \vdash A^{X \longrightarrow Y} \quad \Gamma_1, A^{X \longrightarrow Y}, \Gamma_2 \vdash \Delta}{\Gamma_1, \Gamma, \Gamma_2 \vdash \Delta}\ (cut_2)$$

$$\frac{\Gamma \vdash A^{X \longrightarrow Y}, \Delta_2 \quad \Gamma_1, A^{X \longrightarrow Y} \vdash \Delta}{\Gamma_1, \Gamma \vdash \Delta, \Delta_2}\ (cut_3) \qquad \frac{\Gamma \vdash \Delta_1, A^{X \longrightarrow Y} \quad A^{X \longrightarrow Y}, \Gamma_2 \vdash \Delta}{\Gamma, \Gamma_2 \vdash \Delta_1, \Delta}\ (cut_4)$$

Note: These are just the four "planar" variants of the usual cut rule.

taposition on the left can be interpreted as ordinary relational composition

$$R \otimes S = \{\langle x, z\rangle \mid \exists y. \langle x, y\rangle \in R \wedge \langle y, z\rangle \in S\}$$

while the juxtaposition on the right may be interpreted as the "dual" of the above relational composition:

$$R \oplus S = \{\langle x, z\rangle \mid \forall y. \langle x, y\rangle \in R \vee \langle y, z\rangle \in S\}.$$

This, with proofs given by inclusion, provides a valid interpretation for our logic of generalized relations.

While, from the categorical perspective, this interpretation does not tell us much about the structure of the proofs (since all proofs are identified) it certainly does indicate that a simple interpretation in a bicategory in which the two juxtapositions are identified will not capture the intended structure. Thus there is a requirement for a new categorical structure in which such logics can be interpreted: the new structure is called a *linear* bicategory. A detailed definition will be given later (Definition 2.1); for the moment, we shall consider some other examples.

Another situation where there are two types of composition is given as follows. Imagine planning a trip in which one will travel from place to place by various means of transport. An important decision to make is where the overnight stops will be: this will affect the cost of travel. Figure 1 shows some data that may be used in planning a trip through part of Kenya; the 0-cells are towns, the 1-cells represent means of travel and the corresponding cost, the 2-cells represent cost comparison. The two compositions of 1-cells are $\otimes$ = composition without stopover, and $\oplus$ = composition with stopover. The cost of the $\bot$ unit is the money saved by not staying overnight in a destination. For example, here

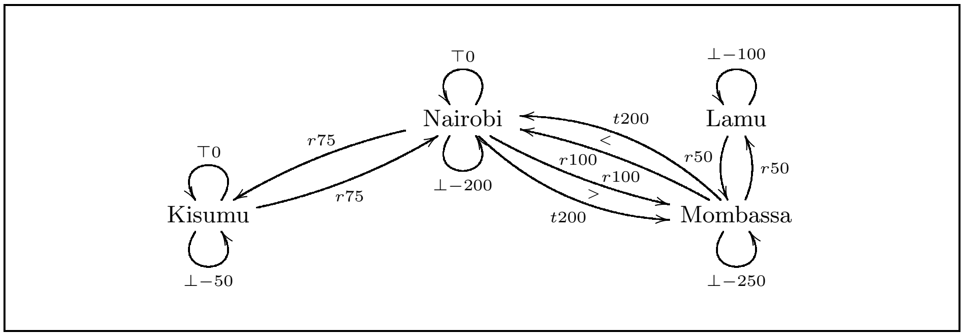


Fig. 1. Travels in Kenya

are two possible ways to get from Nairobi to Lamu: $t200 \otimes r50$, "take the train, then by road, no stopover", (\$250); or $t200 \oplus r50$, "take the train, stopover in Mombassa, then by road" (\$500). Such data has the structure of a linear bicategory.

Notice that while there is still at most one 2-cell between any two 1-cells, this time the other composition, the par, is not simply a de Morgan dual of the usual tensor composition. While in many examples it is the case that the two compositions arise as de Morgan duals, it is important to realize that this is not the only way in which a second composition can be introduced.

A further, nontrivial, example of this is given by the (finite) bicompletion (Hu–Joyal 1997, and other papers cited therein) of a bicategory: if one takes any bicategory and finitely bicompletes the morphism categories, then the original composition produces two different compositions. The resulting structure is a linear bicategory. The two compositions cannot usually be obtained from each other as de Morgan duals and furthermore there will, in general, be many 2-cells between any two 1-cells.

A final simple example is given by the matrices of a $*$-autonomous category (Barr 1979) with products and coproducts (or more generally, of a linearly distributive category with linear products and coproducts (Cockett–Seely 1998)). Given such a category $\mathbf{A}$, we form a bicategory Matrix($\mathbf{A}$) as follows. The 0-cells are the natural numbers. A 1-cell, or "generalized relation", from n to m is an $m \times n$ matrix of objects:

$$\begin{bmatrix} A_{1,1} & \dots & A_{1,n} \\ \vdots & \dots & \vdots \\ A_{m,1} & \dots & A_{m,n} \end{bmatrix}$$

A 2-cell, or "proof", between these matrices is a matrix of maps

$$\begin{bmatrix} f_{1,1} & \dots & f_{1,n} \\ \vdots & \dots & \vdots \\ f_{m,1} & \dots & f_{m,n} \end{bmatrix} : \begin{bmatrix} A_{1,1} & \dots & A_{1,n} \\ \vdots & \dots & \vdots \\ A_{m,1} & \dots & A_{m,n} \end{bmatrix} \Longrightarrow \begin{bmatrix} B_{1,1} & \dots & B_{1,n} \\ \vdots & \dots & \vdots \\ B_{m,1} & \dots & B_{m,n} \end{bmatrix}$$

where $f_{i,j}\colon A_{i,j} \longrightarrow B_{i,j}$. The tensor composition of these generalized relations is given by the matrix multiplication using $\otimes$ for multiplication of objects and coproduct for addition:

$$[A_{i,j}] \otimes [B_{j,k}] = \left[\sum_j (A_{i,j} \otimes B_{j,k}) \right].$$

The $\oplus$ multiplication is given dually by

$$[A_{i,j}] \oplus [B_{j,k}] = \left[\prod_j (A_{i,j} \oplus B_{j,k}) \right].$$

Then one may show that Matrix($\mathbf{A}$) is a linear bicategory; furthermore, if $\mathbf{A}$ is a symmetric $*$-autonomous category, then Matrix($\mathbf{A}$) has linear adjoints (Definition 3.1), and in fact is $*$-linear (Definition 4.9).

Notice that this construction starts with a category and ends with a linear bicategory. However, (as indeed is the case with the bicompletion construction) we should perhaps view this as a construction on linear bicategories themselves. This, in turn, leads to other interesting questions. For example, in ordinary bicategories, enriched categories may be viewed as "monoids" in the matrix bicategory. It therefore becomes natural to ask whether "linear monads" (Definition 4.13) give an interesting notion of linearly enriched categories — and whether these have a hitherto unexploited or unnoticed significance to mathematics? While it is beyond the scope of this paper to pursue such topics, we would like to emphasize the value of having a formulation at this linear bicategorical level of abstraction. It is not simply that it helps to classify known phenomenon but also that it exposes many conceptual ramifications which are still not properly understood in detail.

2. Linear bicategories

Definition 2.1. (**Linear bicategories**) A linear bicategory $\mathbf{B}$ consists of the following data.

1 A class (of "0-cells") $\mathbf{B}_0$.
2 A category $\mathbf{B}_1$ (with "1-cells" as objects and "2-cells" as morphisms).
3 Functors $D_0, D_1 \colon \mathbf{B}_1 \longrightarrow \mathbf{B}_0$, (regarding $\mathbf{B}_0$ as a discrete category). We denote the full subcategory of $\mathbf{B}_1$ given by $\{f \mid D_0(f) = X, D_1(f) = Y\}$ by $\mathbf{B}(X,Y)$.
4 Functors $\otimes_{X,Y,Z}, \oplus_{X,Y,Z} \colon \mathbf{B}(X,Y) \times \mathbf{B}(Y,Z) \longrightarrow \mathbf{B}(X,Z)$. (We call $\otimes$ the tensor, and $\oplus$ the cotensor or par.)
5 1-cells $\top_X, \bot_X \colon 1 \longrightarrow \mathbf{B}(X,X)$
6 Natural isomorphisms

$$\otimes \times 1 \,;\, \otimes \xRightarrow{a_\otimes} 1 \times \otimes \,;\, \otimes \qquad\qquad 1 \times \oplus \,;\, \oplus \xRightarrow{a_\oplus} \oplus \times 1 \,;\, \oplus$$

expressing the associativity of the tensor and cotensor.
As pasting diagrams:

$$\begin{tikzcd} \mathbf{B}(X,Y) \times \mathbf{B}(Y,Z) \times \mathbf{B}(Z,W) \arrow[r, "\otimes \times 1"] \arrow[d, "1 \times \otimes"'] & \mathbf{B}(X,Z) \times \mathbf{B}(Z,W) \arrow[d, "\otimes"] \arrow[dl, phantom, "\Swarrow a_\otimes"] \\ \mathbf{B}(X,Y) \times \mathbf{B}(Y,W) \arrow[r, "\otimes"'] & \mathbf{B}(X,W) \end{tikzcd}$$

$$\begin{tikzcd}
\mathbf{B}(X,Y)\times\mathbf{B}(Y,Z)\times\mathbf{B}(Z,W) \arrow[r, "\oplus\times 1"] \arrow[d, "1\times\oplus"'] \arrow[dr, phantom, "a_\oplus \Nearrow"] & \mathbf{B}(X,Z)\times\mathbf{B}(Z,W) \arrow[d, "\oplus"] \\
\mathbf{B}(X,Y)\times\mathbf{B}(Y,W) \arrow[r, "\oplus"'] & \mathbf{B}(X,W)
\end{tikzcd}$$

7 Natural isomorphisms expressing that $\top$ and $\perp$ are the units for the tensor and cotensor, respectively.

$$1 \xRightarrow{u^L_\otimes} \langle \top_X, 1\rangle\,;\otimes \qquad\qquad \langle \perp_X, 1\rangle\,;\oplus \xRightarrow{u^L_\oplus} 1$$

$$1 \xRightarrow{u^R_\otimes} \langle 1, \top_Y\rangle\,;\otimes \qquad\qquad \langle 1, \perp_Y\rangle\,;\oplus \xRightarrow{u^R_\oplus} 1$$

As pasting diagrams:

$$\begin{tikzcd}
\mathbf{B}(X,X)\times\mathbf{B}(X,Y) \arrow[dr, "\otimes"'] & \mathbf{B}(X,Y) \arrow[l, "\langle\top_X,1\rangle"'] \arrow[r, "\langle 1,\top_Y\rangle"] \arrow[d, "1"] \arrow[dl, phantom, "u^L_\otimes \Swarrow"] \arrow[dr, phantom, "\Searrow u^R_\otimes"] & \mathbf{B}(X,Y)\times\mathbf{B}(Y,Y) \arrow[dl, "\otimes"] \\
& \mathbf{B}(X,Y) &
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbf{B}(X,X)\times\mathbf{B}(X,Y) \arrow[dr, "\oplus"'] & \mathbf{B}(X,Y) \arrow[l, "\langle\perp_A,1\rangle"'] \arrow[r, "\langle 1,\perp_B\rangle"] \arrow[d, "1"] \arrow[dl, phantom, "u^L_\oplus \Nearrow"] \arrow[dr, phantom, "\Nwarrow u^R_\oplus"] & \mathbf{B}(X,Y)\times\mathbf{B}(Y,Y) \arrow[dl, "\oplus"] \\
& \mathbf{B}(X,Y) &
\end{tikzcd}$$

8 Natural transformations (not in general isomorphisms) δ_L and δ_R, called *linear distributivities*.

$$\delta_L\colon 1\times\oplus\,;\otimes \Longrightarrow \otimes\times 1\,;\oplus \qquad\qquad \delta_R\colon \oplus\times 1\,;\otimes \Longrightarrow 1\times\otimes\,;\oplus$$

As pasting diagrams:

$$\begin{tikzcd}
\mathbf{B}(X,Y)\times\mathbf{B}(Y,Z)\times\mathbf{B}(Z,W) \arrow[r, "1\times\oplus"] \arrow[d, "\otimes\times 1"'] \arrow[dr, phantom, "\Swarrow \delta_L"] & \mathbf{B}(X,Y)\times\mathbf{B}(Y,W) \arrow[d, "\otimes"] \\
\mathbf{B}(X,Z)\times\mathbf{B}(Z,W) \arrow[r, "\oplus"'] & \mathbf{B}(X,W)
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbf{B}(X,Y)\times\mathbf{B}(Y,Z)\times\mathbf{B}(Z,W) \arrow[r, "\oplus\times 1"] \arrow[d, "1\times\otimes"'] \arrow[dr, phantom, "\delta_R \Nearrow"] & \mathbf{B}(X,Y)\times\mathbf{B}(Y,W) \arrow[d, "\otimes"] \\
\mathbf{B}(X,Z)\times\mathbf{B}(Z,W) \arrow[r, "\oplus"'] & \mathbf{B}(X,W)
\end{tikzcd}$$

These must satisfy several coherence conditions:

— The usual diagrams to make $\otimes$ a bicategorical composition. These may be found in many references on bicategories; see (Koslowski 1997) for example.

— The usual diagrams to make $\oplus$ a bicategorical composition. These are the analogous diagrams to those above.

— The diagrams (as given in (Cockett–Seely 1992)) expressing the linear distributivity of the tensor over the cotensor. These may be found in (Cockett–Seely 1992).

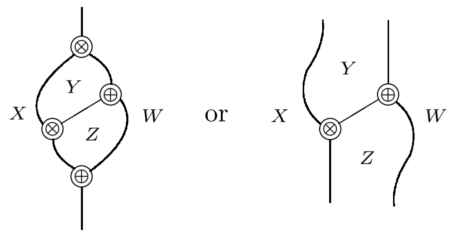
The coherence diagrams for linear distributivity may be less familiar than those for a bicategory, so we shall list a representative sample. The others are given by the evident dualities.

$$\begin{tikzcd}
A \otimes (B \oplus \bot) \arrow[r, Rightarrow, "1 \otimes u_\oplus^R"] \arrow[d, Rightarrow, "\delta_L"'] & A \otimes B \\
(A \otimes B) \oplus \bot \arrow[ur, Rightarrow, "u_\oplus^R"']
\end{tikzcd}
\qquad
\begin{tikzcd}
(A \otimes B) \otimes (C \oplus D) \arrow[r, Rightarrow, "a_\otimes"] \arrow[dd, Rightarrow, "\delta_L"'] & A \otimes (B \otimes (C \oplus D)) \arrow[d, Rightarrow, "1 \otimes \delta_L"] \\
& A \otimes ((B \otimes C) \oplus D) \arrow[d, Rightarrow, "\delta_L"] \\
((A \otimes B) \otimes C) \oplus D \arrow[r, Rightarrow, "a_\otimes \oplus 1"'] & (A \otimes (B \otimes C)) \oplus D
\end{tikzcd}$$

$$\begin{tikzcd}
(A \otimes (B \oplus C)) \otimes D \arrow[rr, Rightarrow, "a_\otimes"] \arrow[d, Rightarrow, "\delta_L \otimes 1"'] & & A \otimes ((B \oplus C) \otimes D) \arrow[d, Rightarrow, "1 \otimes \delta_R"] \\
((A \otimes B) \oplus C) \otimes D \arrow[dr, Rightarrow, "\delta_R"'] & & A \otimes (B \oplus (C \otimes D)) \arrow[dl, Rightarrow, "\delta_L"] \\
& (A \otimes B) \oplus (C \otimes D) &
\end{tikzcd}$$

Note the symmetry in this definition: reversing the 2-cells (giving the *converse* bicategory, $\mathbf{B}^{\mathrm{co}}$) produces a linear bicategory with the tensor and cotensor interchanged. Reversing the compositions $\otimes$ and $\oplus$ (simultaneously), and so also flipping the associativity, unit, and distributivity transformations, produces another linear bicategory, the *dual* or *opposite* bicategory $\mathbf{B}^{\mathrm{op}}$.

Remark 2.2. A key tool that we have used in previous papers (Blute *et al.* 1996; Blute *et al.* 1996a; Cockett–Seely 1997a; Cockett–Seely 1998; Blute *et al.* 1998) is the representation of morphisms in various doctrines of linearly distributive categories (and the representation of derivations in the corresponding logics) by graphs, which we call "circuit diagrams". The reader is referred to the papers cited above for a complete discussion of these "circuits"; a summary may be found in the Appendix. However, we must point out that our usage does differ in inessential ways from the use of string diagrams in (Koslowski 1997), which is based more closely on the conventions of Joyal and Street (Joyal–Street 1991; Street 1995; Street 1996). In particular, note that our circuit diagrams are to be read top to bottom, as is usual with proof nets, and with natural deduction, upon which proof nets are based, and not bottom to top, which is the convention of (Joyal–Street 1991). Also note that in (Koslowski 1997) many string diagrams leave the tensoring of input and outputs to be done implicitly—we too may do that, but since we have two tensors (two "horizontal" compositions) we must be careful about this. In particular, parallel input wires (at the top of the circuit) are to be tensored with the tensor $\otimes$, whereas parallel output wires (at the bottom of the circuit) must be tensored with the cotensor (par) $\oplus$. For example, the linear distributivity δ_L may be represented by either of these circuits:



Example 2.3. We have seen several examples in Section 1.1; here we record some other sources of examples:

1 Recall the motivational example Rel of relations on Sets, in which the two compositions were taken to be relational composition and its dual. This interpretation can be extended to Idl = Mod(Rel), the module bicategory of Rel. The 0-cells of Idl are monads, *i.e.* sets with a reflexive transitive relation (preordered sets). The 1-cells are relations between the underlying sets which are closed under composition with the preorders at each end. That is, $R\colon \langle X, \leq \rangle \longrightarrow \langle Y, \leq \rangle$ is a 1-cell provided $\langle x, y \rangle \in R$ whenever either $x \leq x'$ and $\langle x', y \rangle \in R$ or $y' \leq y$ and $\langle x, y' \rangle \in R$. The 2-cells are given by inclusions as usual. All the compositions are as in Rel.
Now Idl is a closed bicategory (by construction in fact). But furthermore it also inherits the dual composition. To see this suppose $a \leq a'$ and $\langle a', c \rangle \in R \oplus S$; then we wish to conclude that $\langle a, c \rangle \in R \oplus S$. Pick any b in the intermediate set; we know either $\langle a', b \rangle \in R$ or $\langle b, c \rangle \in S$, but the former implies $\langle a, b \rangle \in R$. The unit for this composition at $\langle A, \leq \rangle$ is actually $\not\geq$. In fact, this is a $*$-linear bicategory with $R \Vdash\!\!\dashv \neg R^\circ$ (Section 4.1).

2 The above example can be generalized in the following direction: if **A** is any allegory (Freyd–Scedrov 1990) whose homsets are Boolean algebras, then **A** can canonically be regarded as a $*$-linear bicategory. The other composition is given by $A \oplus B = \neg(\neg B^\circ \otimes \neg A^\circ)^\circ$.

3 By identifying the two compositions, any bicategory can be regarded as a degenerate (or "compact", in the sense $\otimes = \oplus$) linear bicategory.

4 The "suspension" (Street 1995; Street 1996) of a linearly distributive category can be regarded in the obvious manner as a linear bicategory with a single 0-cell.

5 The syntactic linear bicategory of circuits on a "frame" of 0-cells, 1-cells and 2-cells forms a "free" linear bicategory essentially as described in (Blute *et al.* 1996).

2.1. *Linear functors*

What is the appropriate notion of homomorphism of linear bicategories? We shall sketch a candidate in this section, based on the notion of linear functor between linearly distributive categories (Cockett–Seely 1998). This notion generalizes the notion of a monoidal functor between $*$-autonomous categories, and was shown in (Cockett–Seely 1998) to capture much of the structure of linear logic.

Since linear bicategories have two linked compositions, the corresponding notion of homomorphism must necessarily be somewhat complex. The reward, however, is that once the appropriate coherence is embodied in the notion of homomorphism, particular instances of linear bicategories with structure may be derived from the general structural notion. In this way, for example, the linear logic operations **!**, **?**, product, coproduct, even the tensor and par, can all be viewed as linear structure.

Since linear bicategories generalize bicategories, one of our guiding principles (as suggested above) is that in a closed linear bicategory our notion of linear functor must reduce to a morphism of bicategories. This means we must build in enough "duality" to account for the two compositions, but not so much that when the bicategory already has its own duality we get extraneous structure.

Definition 2.4. (**Linear functors**) Let $\mathbf{B}$ and $\mathbf{B}'$ be linear bicategories. A linear functor $F: \mathbf{B} \longrightarrow \mathbf{B}'$ consists of the following:

1 a function $F: \mathbf{B}_0 \longrightarrow \mathbf{B}'_0$
2 two functors $F_\otimes, F_\oplus: \mathbf{B}_1 \longrightarrow \mathbf{B}'_1$ with $D_i\,;F = F_\otimes\,;D_i = F_\oplus\,;D_i$ for $i = 0, 1$
(so $F_\otimes(X \xrightarrow{A} Y) = F(X) \xrightarrow{F_\otimes(A)} F(Y)$, $F_\oplus(X \xrightarrow{A} Y) = F(X) \xrightarrow{F_\oplus(A)} F(Y)$)
3 2-cells $m_\top: \top_{F_\otimes(X)} \Longrightarrow F_\otimes(\top_X)$ and $n_\bot: F_\oplus(\bot_X) \Longrightarrow \bot_{F_\oplus(X)}$
4 natural transformations, which with $m_\top$ and $n_\bot$ make $F_\otimes$ monoidal (or "lax") with respect to $\otimes$ and $F_\oplus$ comonoidal (or "colax") with respect to $\oplus$

$$\begin{aligned} m_\otimes:&\quad F_\otimes(A)\otimes F_\otimes(B) \Longrightarrow F_\otimes(A\otimes B)\\ n_\oplus:&\quad F_\oplus(A\oplus B) \Longrightarrow F_\oplus(A)\oplus F_\oplus(B)\end{aligned}$$

5 natural transformations (called "linear strengths")

$$\begin{aligned} \nu_\otimes^R:&\quad F_\otimes(A\oplus B) \Longrightarrow F_\oplus(A)\oplus F_\otimes(B)\\ \nu_\otimes^L:&\quad F_\otimes(A\oplus B) \Longrightarrow F_\otimes(A)\oplus F_\oplus(B)\\ \nu_\oplus^R:&\quad F_\otimes(A)\otimes F_\oplus(B) \Longrightarrow F_\oplus(A\otimes B)\\ \nu_\oplus^L:&\quad F_\oplus(A)\otimes F_\otimes(B) \Longrightarrow F_\oplus(A\otimes B)\end{aligned}$$

These must satisfy some coherence conditions, which are spelled out in detail in (Cockett–Seely 1998). Some express that $F_\otimes$ is a monoidal functor, and $F_\oplus$ is comonoidal; we give a representative selection of the others dealing with the linear strengths here, the rest being generated by the obvious dualities.

$$\begin{tikzcd} F_\otimes(\bot\oplus A) \arrow[r, Rightarrow, "F_\otimes(u_\oplus^L)"] \arrow[d, Rightarrow, "\nu_\otimes^R"'] & F_\otimes(A) \\ F_\oplus(\bot)\oplus F_\otimes(A) \arrow[r, Rightarrow, "n_\bot\oplus 1"'] & \bot\oplus F_\otimes(A) \arrow[u, Rightarrow, "u_\oplus^L"'] \end{tikzcd}$$

$$\begin{tikzcd}
F_\otimes((A\oplus B)\oplus C) \arrow[rr, Rightarrow, "F_\otimes(a_\oplus)"] \arrow[d, Rightarrow, "\nu_\otimes^R"'] & & F_\otimes(A\oplus(B\oplus C)) \arrow[d, Rightarrow, "\nu_\otimes^R"] \\
F_\oplus(A\oplus B)\oplus F_\otimes(C) \arrow[d, Rightarrow, "n_\oplus\oplus 1"'] & & F_\oplus(A)\oplus F_\otimes(B\oplus C) \arrow[d, Rightarrow, "1\oplus\nu_\otimes^R"] \\
(F_\oplus(A)\oplus F_\oplus(B))\oplus F_\otimes(C) \arrow[rr, Rightarrow, "a_\oplus"'] & & F_\oplus(A)\oplus(F_\oplus(B)\oplus F_\otimes(C))
\end{tikzcd}$$

$$\begin{tikzcd}
F_\otimes((A\oplus B)\oplus C) \arrow[rr, Rightarrow, "F_\otimes(a_\oplus)"] \arrow[d, Rightarrow, "\nu_\otimes^L"'] & & F_\otimes(A\oplus(B\oplus C)) \arrow[d, Rightarrow, "\nu_\otimes^R"] \\
F_\otimes(A\oplus B)\oplus F_\oplus(C) \arrow[d, Rightarrow, "\nu_\otimes^R\oplus 1"'] & & F_\oplus(A)\oplus F_\otimes(B\oplus C) \arrow[d, Rightarrow, "1\oplus\nu_\otimes^L"] \\
(F_\oplus(A)\oplus F_\otimes(B))\oplus F_\oplus(C) \arrow[rr, Rightarrow, "a_\oplus"'] & & F_\oplus(A)\oplus(F_\otimes(B)\oplus F_\oplus(C))
\end{tikzcd}$$

$$\begin{tikzcd}
F_\otimes(A)\otimes F_\otimes(B\oplus C) \arrow[r, Rightarrow, "1\otimes\nu_\otimes^R"] \arrow[d, Rightarrow, "m_\otimes"'] & F_\otimes(A)\otimes(F_\oplus(B)\oplus F_\otimes(C)) \arrow[d, Rightarrow, "\delta_L"] \\
F_\otimes(A\otimes(B\oplus C)) \arrow[d, Rightarrow, "F_\otimes(\delta_L)"'] & (F_\otimes(A)\otimes F_\oplus(B))\oplus F_\otimes(C) \arrow[d, Rightarrow, "\nu_\oplus^R\oplus 1"] \\
F_\otimes((A\otimes B)\oplus C) \arrow[r, Rightarrow, "\nu_\otimes^R"'] & F_\oplus(A\otimes B)\oplus F_\otimes(C)
\end{tikzcd}$$

$$\begin{tikzcd}
F_\otimes(A)\otimes F_\otimes(B\oplus C) \arrow[r, Rightarrow, "1\otimes\nu_\otimes^L"] \arrow[d, Rightarrow, "m_\otimes"'] & F_\otimes(A)\otimes(F_\otimes(B)\oplus F_\oplus(C)) \arrow[d, Rightarrow, "\delta_L"] \\
F_\otimes(A\otimes(B\oplus C)) \arrow[d, Rightarrow, "F_\otimes(\delta_L)"'] & (F_\otimes(A)\otimes F_\otimes(B))\oplus F_\oplus(C) \arrow[d, Rightarrow, "m_\otimes\oplus 1"] \\
F_\otimes((A\otimes B)\oplus C) \arrow[r, Rightarrow, "\nu_\otimes^L"'] & F_\otimes(A\otimes B)\oplus F_\oplus(C)
\end{tikzcd}$$

The key fact about linear functors that we need is that they preserve linear adjoints (Section 3.1) in the following sense: if $A \dashv\!\!\!\vdash B$ then $F_\oplus(A) \dashv\!\!\!\vdash F_\otimes(B)$ and $F_\otimes(A) \dashv\!\!\!\vdash F_\oplus(B)$.

3. Linear Adjunctions and closed linear bicategories

The notion of a complement is fundamental in proof theory. An important distinguishing feature of linear logic is that, unlike classical logic, the introduction of complementation (or negation) does not collapse the proof theory (in the sense that all proofs are equivalent). Indeed the extension of the proof theory from a negation-free fragment to one with negation is full and faithful. From a categorical perspective, this is the transition from linearly distributive categories to $*$-autonomous categories (as described in (Blute *et al.* 1996)).

The consideration of non-symmetric tensors is rather natural when one views them as providing a composition: it was this motivation that led us to the present bicategorical setting. However, one may then ask what is the appropriate notion of negation for such a setting. The answer makes explicit a very elegant categorical aspect of negation, namely that complements are *linear* adjoints. But it also underlines a less comfortable reality, already recognized in the literature, (Barr 1995; Yetter 1990; Lambek 1993; Rosenthal 1992), which we shall discuss in the next section: in noncommutative settings there are various different strengths of negation which must be considered.

Many of the results in this section are straightforward generalizations of results from (Cockett–Seely 1997; Blute *et al.* 1996; Blute *et al.* 1996a; Cockett–Seely 1997a; Cockett–Seely 1998). If a proof is missing here, the reader is referred to these papers.

3.1. *Linear adjunctions*

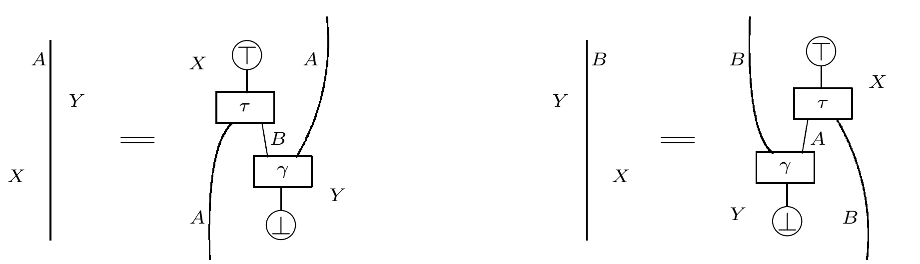
Definition 3.1. (**Linear adjunction**) A linear adjunction $v = \langle \tau, \gamma \rangle : A \dashv\!\!\mid B : X \longrightarrow Y$ ("A is a left linear adjoint to B") consists of the following.

1 1-cells $A : X \longrightarrow Y$, $B : Y \longrightarrow X$.

2 2-cells $\top_X \overset{\tau}{\Longrightarrow} A \oplus B$, $B \otimes A \overset{\gamma}{\Longrightarrow} \bot_Y$, (the "unit" and "counit" respectively), satisfying

$$A \overset{1}{\Longrightarrow} A =$$
$$A \overset{u_\otimes^L}{\Longrightarrow} \top \otimes A \overset{\tau\otimes 1}{\Longrightarrow} (A \oplus B) \otimes A \overset{\delta_R}{\Longrightarrow} A \oplus (B \otimes A) \overset{1\oplus\gamma}{\Longrightarrow} A \oplus \bot \overset{u_\oplus^R}{\Longrightarrow} A$$
$$B \overset{1}{\Longrightarrow} B =$$
$$B \overset{u_\otimes^R}{\Longrightarrow} B \otimes \top \overset{1\otimes\tau}{\Longrightarrow} B \otimes (A \oplus B) \overset{\delta_L}{\Longrightarrow} (B \otimes A) \oplus B \overset{\gamma\oplus 1}{\Longrightarrow} \bot \oplus B \overset{u_\oplus^L}{\Longrightarrow} B$$

As circuits, these equations say:



Notation: Adopting the conventions of bilinear logic, we shall denote the left linear adjoint of a 1-cell B by $B^\perp$, and the right linear adjoint of A by ${}^\perp A$. So this means we have 2-cells $\top \overset{\tau}{\Longrightarrow} B^\perp \oplus B$, $B \otimes B^\perp \overset{\gamma}{\Longrightarrow} \bot$, and $\top \overset{\tau}{\Longrightarrow} A \oplus {}^\perp A$, ${}^\perp A \otimes A \overset{\gamma}{\Longrightarrow} \bot$.

The following equivalences should make this notion somewhat more familiar.

Lemma 3.2. Given 1-cells $A : X \longrightarrow Y$, $B : Y \longrightarrow X$ in a linear bicategory, the following are equivalent.

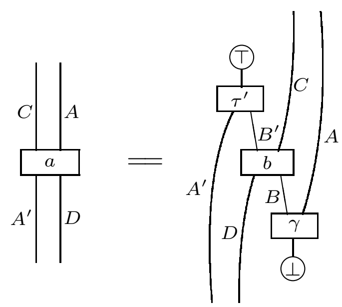
1 A is left linear adjoint to B.

2 For all 0-cells Z, the functor $(\cdot) \otimes A : \mathrm{Hom}(Z, X) \longrightarrow \mathrm{Hom}(Z, Y)$ is left adjoint to the functor $(\cdot) \oplus B$.

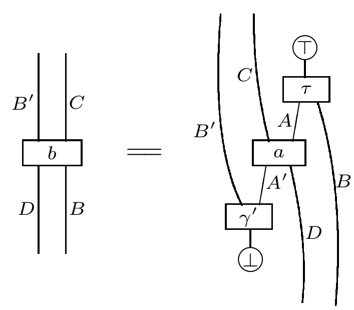
3 For all 0-cells Z, the functor $B \otimes (\cdot) : \mathrm{Hom}(X, Z) \longrightarrow \mathrm{Hom}(Y, Z)$ is left adjoint to the functor $A \oplus (\cdot)$.

Lemma 3.3. (*Australian "mates"*) In a linear bicategory, given 1-cells $A: X \longrightarrow Y$, $B: Y \longrightarrow X$, $A': W \longrightarrow Z$, $B': Z \longrightarrow W$, $C: W \longrightarrow X$, $D: Z \longrightarrow Y$, and linear adjoints $A \dashv\!\!\vdash B$, $A' \dashv\!\!\vdash B'$, there is a bijection between morphisms $C \otimes A \overset{a}{\Longrightarrow} A' \oplus D$ and morphisms $B' \otimes C \overset{b}{\Longrightarrow} D \oplus B$. Moreover, this bijection preserves the compositions $\otimes, \oplus$.

Proof. (Sketch) We want to show, for example, that given $C \otimes A \overset{a}{\Longrightarrow} A' \oplus D$ there is a unique $B' \otimes C \overset{b}{\Longrightarrow} D \oplus B$ so that



b may be defined as follows:



By "preservation of compositions", we mean the following. If we have linear adjunctions $A \dashv\!\!\vdash B$, $A_1 \dashv\!\!\vdash B_1$ (appropriately typed), then there is a linear adjunction $A_1 \otimes A \dashv\!\!\vdash B \oplus B_1$ (and similarly for the case with $\otimes$ and $\oplus$ switched). If furthermore there are linear adjunctions $A' \dashv\!\!\vdash B'$ and $A'_1 \dashv\!\!\vdash B'_1$, inducing $A'_1 \otimes A' \dashv\!\!\vdash B' \oplus B'_1$, and if there are morphisms $A \overset{a}{\Longrightarrow} A' \oplus D$ and $C_1 \otimes A_1 \overset{a_1}{\Longrightarrow} A'_1$, with mates $B' \overset{b}{\Longrightarrow} D_1 \oplus B$ and $B'_1 \otimes C_1 \overset{b_1}{\Longrightarrow} B_1$, then the mate of $a_1 \otimes a \,;\, \delta_L : C_1 \otimes (A_1 \otimes A) \Longrightarrow (A'_1 \otimes A') \oplus D$ is $\delta_R \,;\, b \oplus b_1 : (B' \oplus B'_1) \otimes C \Longrightarrow D_1 \oplus (B \oplus B_1)$. This is essentially proved in (Cockett–Seely 1997a), and is a straightforward exercise in any event, so will be left to the reader. (But note the usual "de Morgan" relationship between $\otimes$ and $\oplus$. And note also that the "planarity" restriction for the noncommutative tensors requires dropping the "parameters" C and D_1 that one might expect in the mates.) $\square$

Corollary 3.4.

1 Any two right (respectively left) linear adjoints to a 1-cell A (respectively B) are isomorphic, in the sense that there is a unique 2-cell which mediates the isomorphism.

2 $\otimes$-composites and $\oplus$-composites of right (respectively left) linear adjoints are right (respectively left) linear adjoints. (In other words, linear adjoints are closed under tensor and par.) Furthermore, $\top \dashv\!\!\mid \bot$ and $\bot \dashv\!\!\mid \top$.

3.2. *Linear homs and closed linear bicategories*

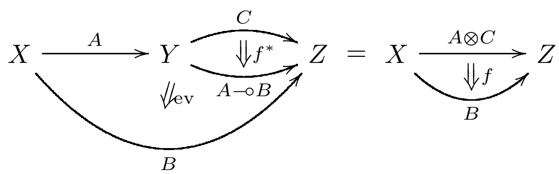
Ordinary adjunctions can be characterized by the fact that they are special (Kan) extensions. We shall see here that linear adjoints admit a similar characterization, in terms of an appropriate notion of "linear extension", which we shall call "linear homs". In addition, this gives a notion of "(linearly) closed structure", and so allows us to give a possible bicategorical analogue to noncommutative $*$-autonomous categories which we shall call "closed linear bicategories"; we shall see another such analogue later. But first we recall some standard notions from the theory of bicategories (although we shall modify the terminology somewhat).

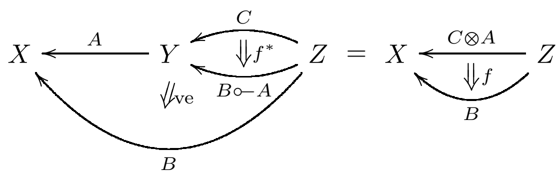
Definition 3.5. (**Left and right homs**) In a bicategory $\mathbf{B}$, given 1-cells $A\colon X \longrightarrow Y$ and $B\colon X \longrightarrow Z$, a *right hom* from A to B is a 1-cell $A \multimap B\colon Y \longrightarrow Z$ together with a 2-cell $A \otimes (A \multimap B) \xRightarrow{\text{ev}} B$ satisfying the following universal property: for any 1-cell $C\colon Y \longrightarrow Z$, and any 2-cell $A \otimes C \xRightarrow{f} B$ there is a unique 2-cell $C \xRightarrow{\text{curry}(f)} A \multimap B$ so that $\text{curry}(f)\ ;\ \text{ev} = f$. The right hom $\langle A \multimap B, \text{ev}\rangle$ is *preserved* by a 1-cell $D\colon Z \longrightarrow W$, if $(\langle A \multimap B) \otimes D, \text{ev} \otimes D\rangle$ is a right hom as well. We call $\langle A \multimap B, \text{ev}\rangle$ *absolute*, if it is preserved by any 1-cell with domain Z.
Left homs, *right cohoms*, and *left cohoms* in $\mathbf{B}$ are just right homs in $\mathbf{B}^{\text{op}}$, $\mathbf{B}^{\text{co}}$ and $\mathbf{B}^{\text{co op}}$, respectively.

A left hom to $B\colon Z \longrightarrow X$ from $A\colon Y \longrightarrow X$, will be denoted $\langle B \circ\!\!- A, \text{ve}\rangle$, where $\text{ve}\colon (B \circ\!\!- A) \otimes A \Longrightarrow B$. The universal property of a left hom induces a bijection between 2-cells $C \Longrightarrow B \circ\!\!- A$ and $C \otimes A \Longrightarrow B$.

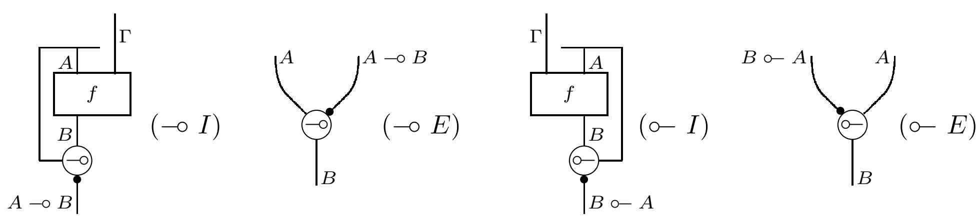
Terminology: We have modified the usual terminology from bicategories here, adopting instead a terminology based on the conventions of Lambek's bilinear logic. The following translation should help the reader more familiar with the bicategorical usage. Our right hom corresponds to the notion of right extension; left homs are right liftings; right cohoms are left extensions; and left cohoms are left liftings. Our usage reflects the logical dualities, so the homs are both adjoints to tensor, and the cohoms are adjoints to par. So in this setting, the "right-left" distinction is determined by reversing 1-cells and interchanging the compositions tensor and par ($\mathbf{B}$ *vs.* $\mathbf{B}^{\text{op}}$ for example), while the "hom-cohom" distinction is determined by reversing 2-cells ($\mathbf{B}$ *vs.* $\mathbf{B}^{\text{co}}$ for example).

If we abbreviate curry(f) by f^*, these pasting diagrams indicate the universal properties for right and left homs.





Translating the above into the language of circuit diagrams requires some care, since this language was designed to accommodate two 1-cell compositions. One obtains the following four new links, together with new rewrites.



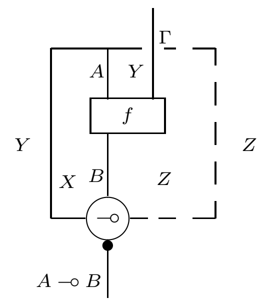
We have omitted the typing information, *i.e.* the labelling of the regions in the links above, but as an example, we shall display the ($\multimap$ I) link with complete typing below. However, first we ought to explain some of the conventions used in the links above.

The boxes labelled f must be valid (sequential) circuits. These may have arbitrarily many inputs, but it is important to note that they may only have the one output as shown. (We shall relax this restriction below, when we define "linear homs".) This restriction corresponds to a similar restriction in intuitionistic linear logic, as discussed in (Cockett–Seely 1997a) in the categorical context; the extension to the present bicategorical setting is straightforward. Relaxing the restriction amounts to making the context $*$-autonomous rather than intuitionistic.

The box f is contained in a "scope box" (only about half of which is shown in the circuit above) which then replaces an input A and an output B with an output $A \multimap B$ (or $B \multimapinv A$ as appropriate). If necessary we indicate the full scope box (as illustrated below), otherwise we abbreviate it as above.

To aid the reader in determining which "port" of these "implication links" is the principal one, we have decorated it with a bullet or black dot.

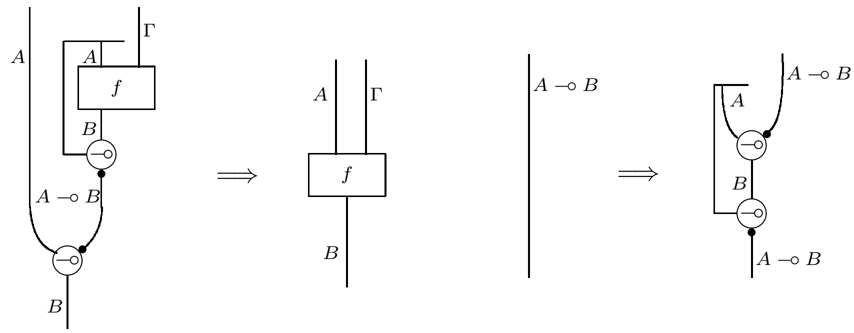
As promised, we present a completely labelled version of the ($\multimap$ I) link, with its complete scope box.

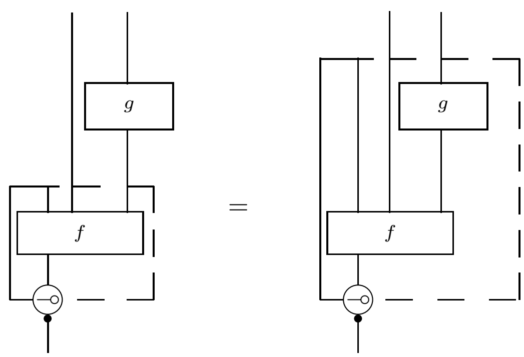


There are some subtle aspects to the labelling above, so some brief discussion may be in order. The reader ought to compare the circuit diagram to the pasting diagram above, to help see how the typing works. First, note that we have labelled the wires A, B, $A \multimap B$, and Γ (representing a finite collection of wires, but corresponding to C in the pasting diagram above). These are the 1-cells. We have labelled the regions between the wires X, Y, Z: these are the 0-cells, which we may regard as the types of the "formulas" given by the 1-cells. The 2-cell is the box labelled f (and the complete link itself, of course—this would be the 2-cell f^* of the pasting diagram). Next, notice that the "scope box" has an "opaque" side, which separates the region Y from the region X, and a "transparent" side, which therefore does not separate the plane into distinct regions (which is why they are both labelled Z: they are the same region). As we suggested in (Cockett–Seely 1997a), the "opaque" side of the box ought to be regarded as the wire A bent to join the $\multimap$ node at the bottom of the box. The "transparent" side of the box does not correspond to any wire, and only is used to mark the subnet which is being bound by the currying process which the ($\multimap$ I) rule represents.

The reader will be able to fill in the missing typing in the other cases, for the links above, and also for the rewrite rules below.

For equivalence of proofs, the necessary rewrites are as follows. Note that the first rewrite is the commutativity of the diagram expressing the universal property given above, and the second rewrite guarantees the uniqueness condition. The third rewrite is technical: it allows the scope boxes to be expanded and contracted. This is essentially a naturality condition.





There are dual rewrites for $\multimap$.

In general, there are also notions of left and right cohoms with respect to the other composition structure, $\oplus$. However, since our intended context ("closed linear bicategories") is "$*$-autonomous" (Remark 3.11), this will be derived structure, given by the usual de Morgan equations. So for example, in the $*$-autonomous context $A \otimes B^{\perp}$ will act as a "co-function space" for $\oplus$, as (in the dual way) $A^{\perp} \oplus B$ is a "function space" for $\otimes$.

From (Street–Walters 1978) we recall the following characterization of ordinary adjunctions.

Proposition 3.6. Given 1-cells $A\colon X \longrightarrow Y$, $B\colon Y \longrightarrow X$, and a 2-cell $\epsilon\colon B \otimes A \Longrightarrow \top_Y$ the following are equivalent:

1 $\langle A, \epsilon\rangle$ is an absolute right hom from B to $\top_Y$.
2 $\langle A, \epsilon\rangle$ is a right hom from B to $\top_Y$ preserved by B;
3 $A \dashv B$ with counit ϵ.

To obtain a similar characterization for linear adjunctions, we just need to consider the preservation and the absoluteness of right homs with respect to $\oplus$ rather than $\otimes$. We first introduce the appropriate terminology.

Definition 3.7. (**Right and left linear homs, closed linear bicategory**) We call a right hom $\langle A \multimap B, \mathrm{ev}\rangle$ *linear*, if it is absolute with respect to $\oplus$, *i.e.*, if $\langle (A \multimap B) \oplus C, \mathrm{ev} \oplus C\rangle$ is a right hom from A to $B \oplus C$, for every 1-cell C composable with B. The linear bicategory **B** *has right linear homs* if it has right homs and they are linear. **B** has left linear homs if $\mathbf{B}^{\mathrm{op}}$ has right linear homs. If **B** has both right and left linear homs, we call **B** a *closed linear bicategory.*

An alternative characterization of linear homs can be given which connects this notion with the notion of tensorial strength which is central to the subject of linear structure. In (Cockett–Seely 1997a) a careful distinction was made between linearly distributive categories with an ordinary closed structure (which we called FILL categories), and those which had the duality we expect of $*$-autonomous categories (which corresponded to the equivalent GILL and BILL categories). This distinction was shown to involve the invertibility of a certain tensorial strength, the inverse giving a costrength. To be more precise, a linear bicategory with left and right homs automatically has a canonical natural transformation $(A \multimap B) \oplus C \Longrightarrow A \multimap (B \oplus C)$, given by linear distributivity and evaluation. This transformation is a tensorial strength; in general it need not be an isomorphism. However, if the hom is absolute, then it will be an isomorphism, in view of the universal property of right homs.

Lemma 3.8. Given 1-cells $A: X \longrightarrow Y$, $B: X \longrightarrow Z$ and $C: Z \longrightarrow W$, and a right hom $\langle A \multimap B, \mathrm{ev}\rangle$, the following are equivalent:

1 $\langle (A \multimap B) \oplus C, \mathrm{ev} \oplus C\rangle$ is a right hom;

2 the canonical 2-cell $(A \multimap B) \oplus C \Longrightarrow A \multimap (B \oplus C)$ is invertible.

Remark: There is a dual result for left homs.

If $Z = X$ and $B = \bot_X$, the isomorphism in the second part is essentially quite a familiar one: the point of Remark 3.11 below is that a linear bicategory with right linear homs is "$*$-autonomous", and so one may identify $A \multimap C$ with $A^\bot \oplus C$ (and dually for $\circ\!-$); under these identifications, the isomorphism mentioned above is essentially given by the associativity of $\oplus$.

Proposition 3.9. Given 1-cells $A: X \longrightarrow Y$, $B: Y \longrightarrow X$, and a 2-cell $\gamma: B \otimes A \Longrightarrow \bot_Y$ the following are equivalent:

1 $\langle A, \gamma\rangle$ is a right linear hom from B to $\bot_Y$;

2 $\langle A, \gamma\rangle$ is a right hom from B to $\bot_Y$ $\oplus$-preserved by B;

3 $A \dashv\!\!\mid B$ with counit γ;

4 $\langle B, \gamma\rangle$ is a left linear hom to $\bot_Y$ from A;

5 $\langle B, \gamma\rangle$ is a left hom to $\bot_Y$ from A $\oplus$-preserved by B.

Proof. Note that the essence of (1) is that $A \cong B \multimap \bot_Y$ and moreover, $A \oplus C \cong B \multimap C$ for any composable C. The essence of (2) is that $A \cong B \multimap \bot_Y$ and moreover, $A \oplus B \cong B \multimap B$. (4) means that $B \cong \bot_Y \circ\!- A$, and that $C \oplus B \cong C \circ\!- A$; (5) means that $B \cong \bot_Y \circ\!- A$, and that $A \oplus B \cong A \circ\!- A$.

(1) $\Rightarrow$ (2) Trivial.

(2) $\Rightarrow$ (3) The identity 2-cell on B induces a 2-cell $\top_X \stackrel{\tau}{\Longrightarrow} A \oplus B$ that is easily seen to be the unit of a linear adjunction.

(3) $\Rightarrow$ (1) "Pasting" with the unit $\top_X \stackrel{\tau}{\Longrightarrow} A \oplus B$ and the counit $B \otimes A \stackrel{\gamma}{\Longrightarrow} \bot_Y$ of the linear adjunction yields the required bijection between 2-cells $B \otimes D \Longrightarrow C$ and $D \Longrightarrow A \oplus C$.

The equivalences with (4), (5) are given by the evident duality. □

Consequently, to obtain a closed linear bicategory, the existence of left and right linear adjoints for every 1-cell is sufficient.

Proposition 3.10. The following are equivalent:

1 **B** is a closed linear bicategory;
2 **B** is a linear bicategory all of whose 1-cells have left and right linear adjoints.

Remark 3.11. A closed linear bicategory **B** is essentially "$*$-autonomous" in the noncommutative sense. For example, we have isomorphisms

$$^{\perp}(A^{\perp} \otimes B^{\perp}) \cong ({}^{\perp}A \otimes {}^{\perp}B)^{\perp} \cong B \oplus A$$

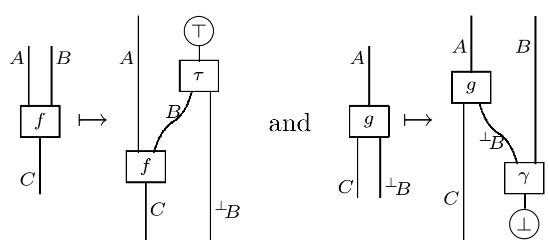
$$^{\perp}(A^{\perp} \oplus B^{\perp}) \cong ({}^{\perp}A \oplus {}^{\perp}B)^{\perp} \cong B \otimes A$$

The essential property is given by the following bijections, which are just the content of Lemma 3.2.

$$\frac{A \otimes B \Longrightarrow C}{\overline{A \Longrightarrow C \oplus {}^{\perp}B}} \qquad \frac{A \otimes B \Longrightarrow C}{\overline{B \Longrightarrow A^{\perp} \oplus C}}$$

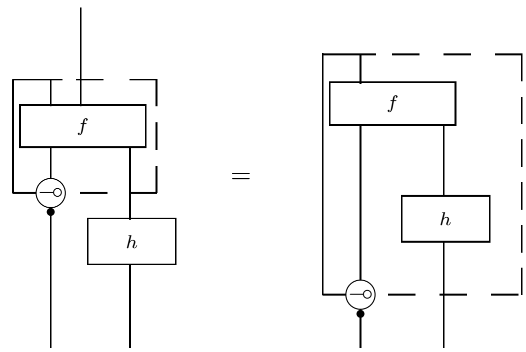
$$\frac{{}^{\perp}A \otimes B \Longrightarrow C}{\overline{B \Longrightarrow A \oplus C}} \qquad \frac{A \otimes B^{\perp} \Longrightarrow C}{\overline{A \Longrightarrow C \oplus B}}$$

From these it is straightforward to show that $(-)^{\perp}$, ${}^{\perp}(-)$ are the required "dualizing" operations. To establish the first bijection (for example), we have the assignments



It is clear these are inverse assignments. Also notice that if we restrict to the endomorphism category $\mathrm{End}(X)$ for any 0-cell X, this is a bilinear category (or noncommutative $*$-autonomous category), in the sense of (Lambek 1993; Cockett–Seely 1997a)).

Finally, let us consider what the existence of linear left and right homs means for our circuit diagrams. It is equivalent to allowing $(-\!\circ\ I)$ and $(\circ\!-\ I)$ to have arbitrarily many outputs exiting the f box. In this case, there must also be a "scope expansion" equivalence to allow a component to be moved into the scope of an introduction rule along an output wire, analogous to the scope expansion rule above, which moved a component along an input wire.



3.3. *The nucleus*

Can we construct a closed linear bicategory from a linear bicategory? The answer to this in fact starts with a linear analogue of a familiar notion, the Map–construction. In this setting we replace ordinary adjoints with linear ones. This does not take us quite far enough, as we shall require 1-cells to have both adjoints—this involves the construction of the *nucleus* of a linear bicategory, paralleling the construction of the nucleus of a linearly distributive (or even a monoidal) category (Cockett–Seely 1998; Rowe 1988; Higgs–Rowe 1989).

From ordinary bicategories the Map-construction is well-known: Map($\mathbf{B}$) has the same 0-cells as $\mathbf{B}$ but only the left adjoint 1-cells. The 2-cells again are the ones of $\mathbf{B}$. By Lemma 3.3 (in the case $\otimes = \oplus$), Map($\mathbf{B}$) may equivalently be defined as saying Map$(\mathbf{B})^{\mathrm{op}}$ has as 1-cells right adjoints. However, in the present linear context, perhaps the best formulation would define Map($\mathbf{B}$), for $\mathbf{B}$ a linear bicategory, as having linear adjoints as 1-cells, and "mates" (pairs of 2-cells, as in Lemma 3.3) for 2-cells.

Recall that any linear bicategory is, with respect to the tensor an ordinary bicategory. Thus, we might ask whether there is a difference between the bicategorical map construction (with respect to tensor) and the linear map construction. In general there is an enormous difference. To see this consider Rel: the bicategorical maps of Rel are just functions; on the other hand, every relation is a linear map.

We are also interested in restricting the 1-cells to those that are both left and right linear adjoint. Call this the first nucleus $\mathrm{Nuc}_1(\mathbf{B})$. Since the right and left linear adjoints to the selected 1-cells need not belong to $\mathrm{Nuc}_1(\mathbf{B})$, we iterate this construction to obtain Nuc($\mathbf{B}$). Notice that iterating the Map-construction only isolates the 1-cells with an infinite chain of (linear) adjoints on the right. If we alternate the Map-construction with its dual that isolates right linear adjoints, we obtain Nuc($\mathbf{B}$) again, whose 1-cells have an infinite chain of linear adjoints on both sides. Nuc($\mathbf{B}$) is a linear bicategory and has the following property.

Proposition 3.12. For a linear bicategory $\mathbf{B}$, Nuc($\mathbf{B}$) is the largest 2-full closed linear sub-bicategory.

3.4. *Dualizing 1-cells*

It is also possible to describe closed linear bicategories as bicategories with left and right homs and "dualizing 1-cells", in the following sense. We shall say of a bicategory with left and right homs that it has *dualizing 1-cells* if for every 0-cell X there is a 1-cell $\bot_X : X \longrightarrow X$ so that for any 1-cell $A : X \longrightarrow Y$, the natural maps $A \Longrightarrow \bot_X \multimapinv (A \multimap \bot_X)$ and $A \Longrightarrow (\bot_Y \multimapinv A) \multimap \bot_Y$ are isomorphisms. If $\mathbf{B}$ is a bicategory with left and right homs and dualizing 1-cells, we may define a par $\oplus$ by $A \oplus B = ((\bot_Y \multimapinv A) \otimes (\bot_Y \multimapinv B)) \multimap \bot_Y$, or by the isomorphic $\bot_X \multimapinv ((A \multimap \bot_X) \otimes (B \multimap \bot_X))$. Some easy (and indeed, well-known) exercises: show these are indeed isomorphic, that the defined $\oplus$ has the appropriate par structure, that there are linear distributivities as required for $\mathbf{B}$ to be a linear bicategory, and finally, that $A \dashv\!\!\mid \bot_Y \multimapinv A$ and $A \multimap \bot_X \dashv\!\!\mid A$. So we have the following alternative characterization.

Proposition 3.13. A bicategory with left and right homs and specified dualizing 1-cells is, together with the induced dual composition, a closed linear bicategory.

In the presence of left and right homs, the existence of dualizing 1-cells can also be formulated in terms of equivalences between hom categories: for any 0-cells X, Y, we must have an equivalence

$$\begin{tikzcd} \mathbf{B}(X,Y) \arrow[r, bend right, "(\cdot)\multimap\bot_X"'] \arrow[r, phantom, "\top"] & \mathbf{B}(Y,X)^{\mathrm{op}} \arrow[l, bend right, "\bot_X\multimapinv(\cdot)"'] \end{tikzcd}$$

This notion is extended by requiring the dualizing 1-cells to be *cyclic* (*cf.* (Yetter 1990)), so that the above equivalence is essentially dual to the corresponding one induced by $\bot_Y$. This suggests the following definition from (Koslowski 1998).

Definition 3.14. (**Cyclic $*$-autonomous bicategory**) A bicategory $\mathbf{B}$ is a *cyclic $*$-autonomous bicategory* if for any X, Y there are adjoint equivalences

$$\begin{tikzcd} \mathbf{B}(X,Y) \arrow[r, bend right, "(\cdot)^*"'] \arrow[r, phantom, "\top"] & \mathbf{B}(Y,X)^{\mathrm{op}} \arrow[l, bend right, "((\cdot)^*)^{\mathrm{op}}"'] \end{tikzcd}$$

and if for every 1-cell $X \xrightarrow{A} Y$ there is a right hom A^* from A to $\top_X^*$ so that these right homs are natural in A.

This requires that the $\top_X^*$ are the dualizing cells $\bot_X$. Furthermore the definition implies that $\mathbf{B}$ is a closed bicategory for which, for any 2-cell $f : A \Longrightarrow B$ in $\mathbf{B}(X,Y)$, $f \multimap \bot_X$ and $\bot_Y \multimapinv f$ coincide. Using the evident de Morgan equations, $\mathbf{B}$ can in fact be equipped with the structure of a closed *linear* bicategory.

This notion is rather stronger than the ones we have been discussing so far, and indeed, many closed linear bicategories are not cyclic $*$-autonomous. It corresponds to cyclic (multiplicative) linear logic, and leads us to the next section, where we shall analyze it further, and to a sequel to this introductory paper, where we shall show how the (generalized) Chu construction connects these notions.

4. Cyclic linear adjoints and linear monads

In linear bicategories there is a stronger notion of adjunction which is left/right symmetric: cyclic linear adjoints. It arises quite naturally in settings which are constructed from symmetric settings such as the linear bicategory of matrices of a symmetric $*$-autonomous category.

Associated with a cyclic linear adjoint is a "linear monad" obtained by composing the 1-cells of the adjoint in the usual manner. A linear monad consists of a monad/comonad pair which satisfies a number of equations. This structure is, in fact, simply the appropriate generalization to the linear setting of Frobenius algebras—or put slightly differently, a linear monad in a (degenerate linear) bicategory is a Frobenius algebra.

Frobenius algebras are also known as ambialgebras which provide a special example of a topological quantum field theory. The connection between these disparate ideas can be sketched as follows: there is a linear functor k from $\mathbf{B}$ to its "enveloping" linear bicategory $k: \mathbf{B} \longrightarrow \widetilde{\mathbf{B}}$ so that any linear functor $F: \mathbf{B} \longrightarrow \mathbf{B}'$ can be expressed as the composite of k with a unique (linear) functor which preserves all the ($\otimes$ and $\oplus$) composition structure on the nose. For the linear bicategory with a single 0-cell, 1-cell, and 2-cell, this enveloping bicategory is a linear bicategory with a generic linear monad. Restricing all this to the compact case gives a generic Frobenius algebra or ambialgebra. These categories, in turn, are known to be equivalent to topological categories (for example $\mathbf{SDiff}^{1+1}$ whose maps are 2-dimensional manifolds with borders); see (Quinn 1995) for more details.

Although we shall not develop the construction here, linear monads can be used as the basis for a "linear module" construction. This construction produces a new linear bicategory in which the linear monads are the 0-cells and the monad and comonad become the identity 1-cells for the two compositions. The new 1-cells are then modules which have compatible actions and coactions at each end in an analogous manner to the standard module construction. (Idl actually provides a degenerate example of this construction, see Example 2.3).

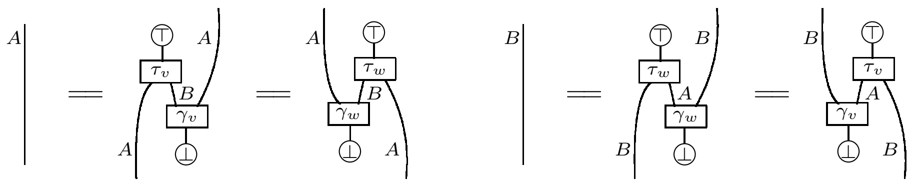
The construction is significant as it provides another way to produce non-degenerate linear bicategories. In particular, even if one starts in a degenerate setting, with Frobenius algebras, the module construction will still produce a highly non-degenerate linear bicategory.

We start our discussion by showing how we may also extract from any linear bicategory a "cyclic nucleus". This is a one step process which, unlike the Map and Nuc constructions above, places a significant "cyclic" constraint on the 2-cells which are carried forward.

4.1. *Cyclic linear adjoints*

Definition 4.1. (**Cyclic linear adjoints**) In a linear bicategory $\mathbf{B}$, a cyclic linear adjoint $\langle v, w\rangle = \langle \tau_v, \gamma_v, \tau_w, \gamma_w\rangle : A \Vdash\!\!\dashv B: X \longrightarrow Y$ is a pair of linear adjoints $v = \langle \tau_v, \gamma_v\rangle : A \dashv\!\!\mid B: X \longrightarrow Y$ and $w = \langle \tau_w, \gamma_w\rangle : B \dashv\!\!\mid A: Y \longrightarrow X$.

In pictures we have therefore two ways of introducing "snakes" into the A and the B wires.



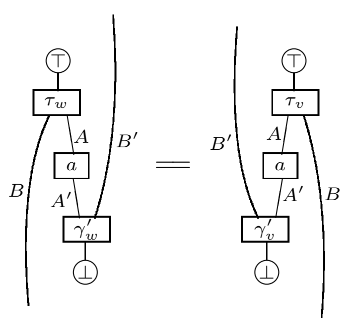
Notice in particular that there is a canonical cyclic linear adjunction $\top \mathrel{\Vert\!\!\vdash\!\!\dashv\!\!\Vert} \bot$ so that we are always guaranteed the presence of certain cyclic linear adjoints. (Whenever we use this cyclic linear adjunction, we shall suppose we are using the canonical one.) Note also that given a cyclic linear adjunction $A \mathrel{\Vert\!\!\vdash\!\!\dashv\!\!\Vert} B$ there is a canonically induced $B \mathrel{\Vert\!\!\vdash\!\!\dashv\!\!\Vert} A$, given by interchanging the two τ's and interchanging the two γ's.

As before we may form a linear bicategory whose 1-cells are these adjoints. However, we must choose the 2-cells a little more carefully. It is, in fact, from this choice that the appropriate coherence conditions can be derived: see (Koslowski 1998) for closed bicategories and (Rosenthal 1994) for noncommutative $*$-autonomous categories.

Definition 4.2. Given a linear bicategory $\mathbf{B}$, the bicategory $\mathrm{CMap}(\mathbf{B})$ is given as follows:

0-Cells: Those of $\mathbf{B}$.
1-Cells: Cyclic linear adjoints $A \mathrel{\Vert\!\!\vdash\!\!\dashv\!\!\Vert} B$.
2-Cells: $\langle a, b\rangle\colon A \mathrel{\Vert\!\!\vdash\!\!\dashv\!\!\Vert} B \Longrightarrow A' \mathrel{\Vert\!\!\vdash\!\!\dashv\!\!\Vert} B'$ where $a\colon A \Longrightarrow A'$ and $b\colon B' \Longrightarrow B$ such that b is the two-way mate of a (*cf.* Lemma 3.3).

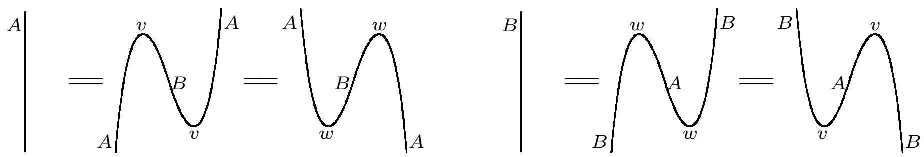
This last condition is equivalent to demanding that a satisfies:



As b is determined by being the mate there is no need to say anything about it, other than the common composite above equals b. Notice that, in general, not every map a will satisfy this condition, so that the maps between cyclic linear adjoints will be a proper subcategory of the category of all the 2-cells between the (say) left adjoints.

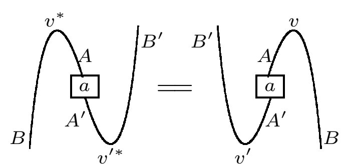
Definition 4.3. (**Cyclic 2-cells**) Given $A \mathrel{\Vert\!\!\vdash\!\!\dashv\!\!\Vert} B$, $A' \mathrel{\Vert\!\!\vdash\!\!\dashv\!\!\Vert} B'$, a 2-cell $a\colon A \Longrightarrow A'$ of $\mathbf{B}$ is cyclic if it satisfies the condition above, so a induces a 2-cell of $\mathrm{CMap}(\mathbf{B})$.

Notation: The reader will have become used to these "snakes", so we shall abbreviate the circuits involving them by using "bends" labelled with the name of the adjunction for the τ's and γ's. So for instance, the defining equations for $A \mathrel{\Vert\!\!\vdash\!\!\dashv\!\!\Vert} B$ will be represented as



It is straightforward to establish that 2-cells in CMap($\mathbf{B}$) can be composed (using the composition of the underlying linear bicategory) and that the 1-cells can also be composed in the two different ways. For example, given $A \Vdash\!\!\dashv B \colon X \longrightarrow Y$, $A' \Vdash\!\!\dashv B' \colon Y \longrightarrow Z$, the tensor of these is $A' \otimes A \Vdash\!\!\dashv B \oplus B'$, and the par is given dually.

Next we shall consider linear adjoints in CMap($\mathbf{B}$). It is easy to see that given a 1-cell in CMap($\mathbf{B}$) (*i.e.* a cyclic linear adjoint $\langle v, w\rangle \colon A \Vdash\!\!\dashv B$), it is canonically a linear adjoint $\bar{v} \colon (A \Vdash\!\!\dashv B) \dashv\!\!\mid (B \Vdash\!\!\dashv A)$ where $\tau_{\bar{v}} = \langle \gamma_w, \tau_v\rangle$ and $\gamma_{\bar{v}} = \langle \tau_w, \gamma_v\rangle$. These are 2-cells in CMap($\mathbf{B}$). This linear adjoint canonically induces a linear adjoint $\bar{v}^* \colon (B \Vdash\!\!\dashv A) \dashv\!\!\mid (A \Vdash\!\!\dashv B)$, given by $\tau_{\bar{v}^*} = \langle \gamma_v, \tau_w\rangle$ and $\gamma_{\bar{v}^*} = \langle \tau_v, \gamma_w\rangle$. Moreover, given another $A' \Vdash\!\!\dashv B'$ and a cyclic $a \colon A \Longrightarrow A'$, there are two evident ways to construct a mate for a: using $\bar{v}$ we obtain a mate we shall call a^*, and using $\bar{v}^*$ we obtain a mate we shall call ${}^*a$. But in fact, these may be shown to be equal.



We can summarize this situation in the following proposition.

Proposition 4.4. CMap($\mathbf{B}$) is a linear bicategory in which each 1-cell A has a cyclic linear adjoint such that the two canonical ways of taking mates are equal.

4.2. $*$-*linear bicategories*

In CMap($\mathbf{B}$) we have seen that we have the ability to "flip" linear adjoints; this is central to the notion of cyclicity in linear bicategories. In this section we shall develop the algebraic properties that permit this in general, with the aim of formulating the notion of a $*$-linear bicategory. We begin with the two basic ways of viewing this "flipping" process.

Proposition 4.5. (**Definition: cyclic mates**) In a linear bicategory $\mathbf{B}$ the following are equivalent:

1 For every linear adjoint $v \colon A \dashv\!\!\mid B$ there is a linear adjoint $v^* \colon B \dashv\!\!\mid A$. For any other $v' \colon A' \dashv\!\!\mid B'$ and any $a \colon A \Longrightarrow A'$, the two canonically constructed mates for a are

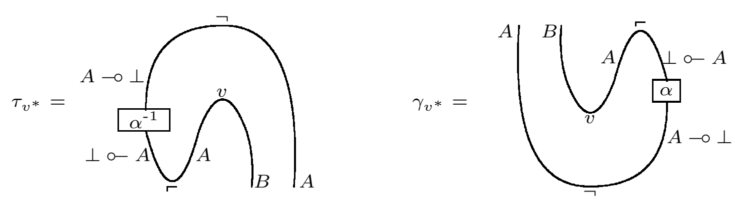
equal, *viz.* a^* constructed using v, v' and ${}^*a$ using v^*, v'^*. We call the linear adjunction v^* a **cyclic mate** of v.

2 For every $A\colon X \longrightarrow Y$, the hom $\bot_Y \multimapinv A$ exists and is linear if and only if the hom $A \multimap \bot_X$ exists and is linear, and there is a natural isomorphism α for all such A: $\alpha\colon \bot_Y \multimapinv (\cdot) \overset{\sim}{\Longrightarrow} (\cdot) \multimap \bot_X$.

Notice that this proposition does not presume that linear adjoints exist for all 1-cells. However, in so far as they do, they are cyclic in the appropriate sense. We shall say that a linear bicategory "has cyclic mates" if it satisfies the conditions of Proposition 4.5.

Proof. (1) $\Rightarrow$ (2) This is immediate upon noting that $A \dashv\!\!\vdash B$ if and only if B is isomorphic to $\bot \multimapinv A$ and $B \dashv\!\!\vdash A$ if and only if B is isomorphic to $A \multimap \bot$. The naturality of the induced isomorphism α amounts to the equality of the mates $\bot \multimapinv a$ and $a \multimap \bot$ of any $a\colon A \Longrightarrow A'$.

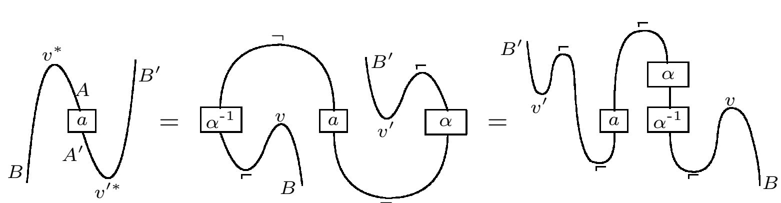
(2) $\Rightarrow$ (1) Given α, we can construct v^* as indicated by these circuits:

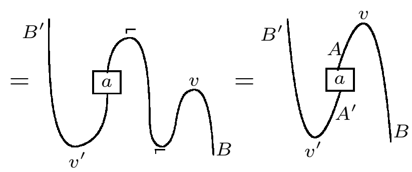


Here, we have denoted the canonical linear adjoints (Propositions 3.13 and 3.9) $A \dashv\!\!\vdash \bot_Y \multimapinv A$ and $A \multimap \bot_X \dashv\!\!\vdash A$ by $\ulcorner$ and $\urcorner$ respectively.

It is worth noting that there is an alternative construction of such a linear adjunction, which we shall denote ${}^*v\colon B \dashv\!\!\vdash A$, obtained by "reflecting" the construction above (under this "reflection", α and α^{-1} change roles). This need not equal v^*, but the proof that it satisfies (1) is similar to what we shall say here. The connection between these two canonical alternatives will be examined in the next proposition. For the moment, let us note that ${}^*(v^*) = v = ({}^*v)^*$. For the rest of this proof, we shall suppose v^* constructed as in the circuits above.

Given $v'\colon A' \dashv\!\!\vdash B'$ and cyclic $a\colon A \Longrightarrow A'$, the two possible mates are the "ends" of the chain of equations below, and their equality is shown (using the definition of v^* at the first step, naturality of α at the second step, composition of α, α^{-1}, and the usual equations for linear adjoints for the rest of the steps).





□

A linear bicategory which has cyclic mates corresponds therefore to a noncommutative logic in which ${}^{\perp}A \cong A^{\perp}$, *i.e.* in which (up to isomorphism) there is only one "dual" of an object (or 1-cell). This is, of course, the basic premise of *cyclic* linear logic (Yetter 1990).

We shall, however, require that this dualizing process satisfies more in two important regards. The first requirement is that the two ways of forming the double dual should be coherently related (Rosenthal 1994), Proposition 4.6 (4). The second demand, to which we shall return later, is that dualization should behave coherently with respect to the two compositions, Proposition 4.7.

We explore the first requirement in the following proposition. We shall use the notation of the proof above, with two cyclic mates ${}^*v$ and v^* for a given linear adjunction v. Note that there is a close connection between the various notions here: for example, given closed structure, one gets a canonical construction of the adjoints *via* $\perp \circ\!\!- A$ and $A \multimap \perp$, and given a canonical way of "flipping" (*i.e.* v^*), then there is a canonical construction of α, and the dual "flipping" (${}^*v$) is then a consequence of all this canonical structure. One of our points is to see what properties α needs to mimic this canonical structure.

Proposition 4.6. In a linear bicategory $\mathbf{B}$ with cyclic mates the following are equivalent:

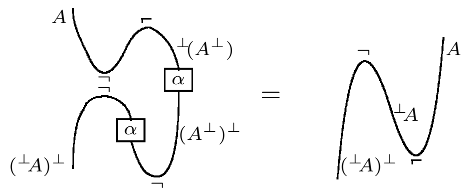
1 For each linear adjoint $v = (v^*)^*$.
2 For each linear adjoint $v = {}^*({}^*v)$.
3 For each linear adjoint ${}^*v = v^*$.
4 The natural equivalence α, as above, satisfies:

$$\begin{tikzcd}
A \arrow[r, Rightarrow, "\eta'"] \arrow[d, Rightarrow, "\eta"'] & (\perp_Y \circ\!\!- A) \multimap \perp_Y \\
\perp_X \circ\!\!- (A \multimap \perp_X) \arrow[r, Rightarrow, "\alpha"'] & (A \multimap \perp_X) \multimap \perp_Y \arrow[u, Rightarrow, "\alpha \multimap \perp_Y"']
\end{tikzcd}$$

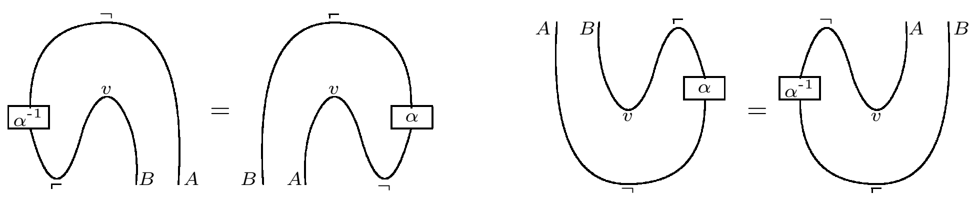
Proof. Recall that ${}^*(v^*) = ({}^*v)^* = v$, so showing (1) if and only if (2) if and only if (3) is a standard exercise in logic.

(3) ⇔ (4) To see this, note that the commutative diagram for cyclic dualizing objects may be expressed as the following circuit equation (where we have used the abbreviations

${}^{\perp}A = \perp \multimapinv A$ and $A^{\perp} = A \multimap \perp$ to make the labels fit more easily).



It is fairly straightforward to show this is equivalent to the following equations, for any $v\colon A \dashv\!\!\dashv B$. To show these for the adjunction $B^{\perp} \dashv\!\!\dashv B$ is a simple use of the adjunction equation, and the naturality of α then gives these more general equations.



But this is exactly $v^{*} = {}^{*}v$. $\square$

Note that if $\mathbf{B}$ is a closed linear bicategory, the conditions of Proposition 4.6 are equivalent to $\mathbf{B}$ being a cyclic $*$-autonomous bicategory (Definition 3.14). In fact, the only difference between these concepts is that for Proposition 4.6 we have not assumed that all linear adjoints exist; it is possible to formulate a "local" version of Definition 3.14 equivalent to Proposition 4.6. The point here is that the condition (from Definition 3.14) that the opposite of the (external) mate of c_{XY} be c_{YX}^{-1} is equivalent to Proposition 4.6 (4).

There is an alternative way of bridging these concepts: note that when a linear bicategory has cyclic mates $\mathrm{Map}(\mathbf{B}) = \mathrm{Nuc}(\mathbf{B})$, so $\mathrm{Map}(\mathbf{B})$ is a closed linear bicategory. Thus to regain the situation where all linear adjoints exist, one need only pass to $\mathrm{Map}(\mathbf{B})$.

We now turn to the question of the interaction of dualizing with the two compositions:

Proposition 4.7. In a linear bicategory $\mathbf{B}$ with cyclic mates the following are equivalent:

1 If $v\colon A \dashv\!\!\dashv B$ and $w\colon A' \dashv\!\!\dashv B'$, then $(v \otimes w)^{*} = w^{*} \oplus v^{*}$ and $(v \oplus w)^{*} = w^{*} \otimes v^{*}$. If $u_R\colon \top \dashv\!\!\dashv \perp$ and $u_L\colon \perp \dashv\!\!\dashv \top$ are the canonical adjunctions (so $u_L = \langle (u_{\oplus}^{L})^{-1}, (u_{\otimes}^{L})^{-1} \rangle$ and u_R dual), then $u_R^{*} = u_L$.

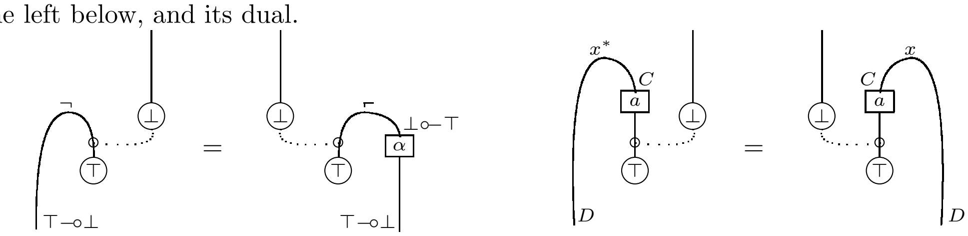
2 If $v\colon A \dashv\!\!\dashv B$ and $w\colon A' \dashv\!\!\dashv B'$, then ${}^{*}(v \otimes w) = {}^{*}w \oplus {}^{*}v$ and ${}^{*}(v \oplus w) = {}^{*}w \otimes {}^{*}v$. Furthermore, ${}^{*}u_R = u_L$, for u_R, u_L as above.

3 α satisfies the following conditions.

$$\begin{tikzcd}
\perp \multimapinv \top \arrow[r, Rightarrow, "d_0"] \arrow[d, Rightarrow, "\alpha_{\top}"'] & \perp \\
\top \multimap \perp \arrow[ur, Rightarrow, "d_1"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\perp \multimapinv (A \otimes A') \arrow[r, Rightarrow, "d_2"] \arrow[d, Rightarrow, "\alpha_{A \otimes A'}"'] & (\perp \multimapinv A') \oplus (\perp \multimapinv A) \arrow[d, Rightarrow, "\alpha_{A'} \oplus \alpha_A"] \\
(A \otimes A') \multimap \perp \arrow[r, Rightarrow, "d_3"'] & (A' \multimap \perp) \oplus (A \multimap \perp)
\end{tikzcd}$$

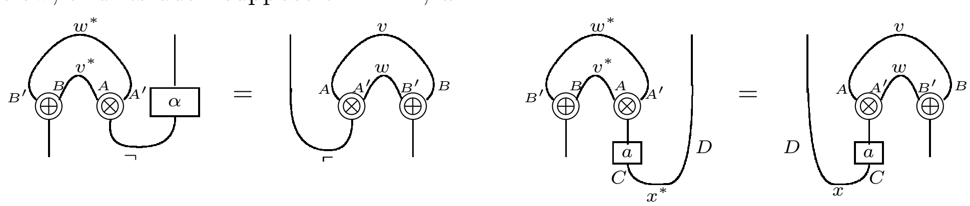
and the duals for $\alpha_\perp$ and $\alpha_{A\oplus A'}$, where d_0, d_1, d_2, d_3 are the canonical isomorphisms.

Proof. To see that (1) $\Leftrightarrow$ (3), it is perhaps simplest to translate the categorical diagrams of (3) into circuit equations. The condition on $\alpha_\top, \alpha_\perp$ is equivalent to the equation on the left below, and its dual.



But this is evidently equivalent to $u_R^* = u_L$. It is useful to note that this circuit equation may be strengthened (using the naturality of α) to the circuit equation on the right above, for any $a\colon C \Longrightarrow \top$ and $x\colon D \dashv\!\mid C$.

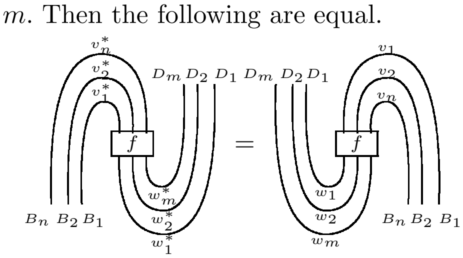
Similarly, the condition on $\alpha_{A\otimes A'}, \alpha_{A\oplus A'}$ amounts to the circuit equation on the left below, and its dual: suppose $v\colon A \dashv\!\mid B$, $w\colon A' \dashv\!\mid B'$.



But this is obviously equivalent to $(v\otimes w)^* = w^* \oplus v^*$. Again, note that the equality above is equivalent to the equation on the right above, and its dual, for any $a\colon A \otimes A' \Longrightarrow C$ and $x\colon D \dashv\!\mid C$.

It is clear that (2) $\Leftrightarrow$ (3) may be shown dually, using α^{-1} instead of α. $\square$

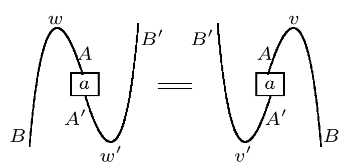
Remark 4.8. In fact, these (equivalent) conditions are also equivalent to a general "circuit flipping" condition: given any linear adjoints, the two canonical ways of "flipping" wires around a component (or 2-cell) must be equal. The cases above (where the component has two inputs, one output, and where it has no inputs and one output, and their duals) are sufficient to generate all other cases. The general situation is pictured below (where we have taken 3 as a generic integer—note that the cases where there are either no inputs or no outputs for f are also to be considered here). Suppose $f\colon A_1 \otimes A_2 \otimes A_3 \otimes \cdots \otimes A_n \Longrightarrow C_1 \oplus C_2 \oplus C_3 \oplus \cdots \oplus C_m$, $v_i\colon A_i \dashv\!\mid B_i$, $w_j\colon D_j \dashv\!\mid C_j$, $i = 1, \ldots, n$, $j = 1, \ldots, m$. Then the following are equal.



Definition 4.9. (**$*$-linear bicategories**) A closed linear bicategory is $*$-linear if it satisfies the (equivalent) conditions of Proposition 4.5, the (equivalent) conditions of Proposition 4.6, and the (equivalent) conditions of Proposition 4.7.

Remark 4.10. Note that being $*$-linear is equivalent to being cyclic $*$-autonomous with the linear structure $\oplus$, $\perp$ defined in the canonical way, as suggested by de Morgan duality and Proposition 4.7. In fact, this shows that a closed linear bicategory that satisfies the conditions of Proposition 4.5 and the conditions of Proposition 4.6 is biequivalent to a $*$-linear bicategory.

Remark 4.11. (**Absolutely $*$-linear bicategories**) We note that CMap($\mathbf{B}$) is a $*$-linear bicategory (in a sense, the canonical such). It may be of interest to compare it with the other construction of a linear bicategory with adjoints, *viz.* Map($\mathbf{B}$). These need not be the same, but we shall end this section with a further condition on a cyclic bicategory which corresponds to Map($\mathbf{B}$) = CMap($\mathbf{B}$). We express this in the following way.
We call a closed linear bicategory that satisfies the conditions of Proposition 4.5 and the conditions of Proposition 4.6 *absolutely $*$-linear* if for any $\langle v, w\rangle\colon A \Vdash\!\!\dashv\!\Vert\, B$, $\langle v', w'\rangle\colon A' \Vdash\!\!\dashv\!\Vert\, B'$, and any $a\colon A \Longrightarrow A'$, the two possible mates of a are equal.

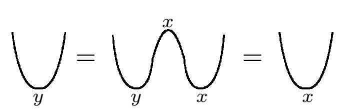


This looks like other conditions we have seen, but note that we have not used the cyclic mates (Proposition 4.5) here, and so there need be no relationship between v, w, v', w'. These may be arbitrary linear adjunctions. In particular, CMap($\mathbf{B}$) will only satisfy this condition if it is equivalent to Map($\mathbf{B}$). But this condition does restrict the choice of cyclic mates in general.

Lemma 4.12. $\mathbf{B}$ is absolutely $*$-linear if and only if cyclic mates are unique.

So absolutely $*$-linear bicategories are $*$-linear.

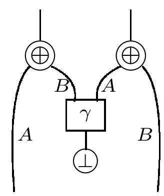
Proof. "If" is easy since uniqueness implies that $v' = w'^*$ and $v = w^*$. "Only if": Let $v\colon A \dashv\!\Vert\, B$ be a linear adjoint, and suppose x and y are two cyclic mates $B \dashv\!\Vert\, A$. Considering $\langle v, x\rangle\colon A \Vdash\!\!\dashv\!\Vert\, B$, $\langle v, y\rangle\colon A \Vdash\!\!\dashv\!\Vert\, B$, and $a\colon A \Longrightarrow A$ the identity, we get $\gamma_y = \gamma_x$ as follows:



Dually we can show $\tau_x = \tau_y$ and so $x = y$. $\square$

4.3. *Linear monads.*

Notice that a linear adjoint $\langle \tau, \gamma \rangle : A \dashv\!\!\dashv B : X \longrightarrow Y$ induces a $\otimes$-monad $\langle A \oplus B, \tau, \mu \rangle$ on X, where μ is given by δ_R ; $1 \oplus \delta_L$; $1 \oplus (\gamma \oplus 1)$; $1 \oplus u^L_\oplus$. As a circuit, this is essentially the following.



Dually there is a $\oplus$-comonad $\langle B \otimes A, \gamma, \delta \rangle$ on Y.

For a cyclic linear adjoint we therefore obtain a $\otimes$-monad and $\oplus$-comonad on the same 0-cell X (as well as on Y), which satisfy some additional coherences on both the domain and codomain. This structure we call a "linear monad".

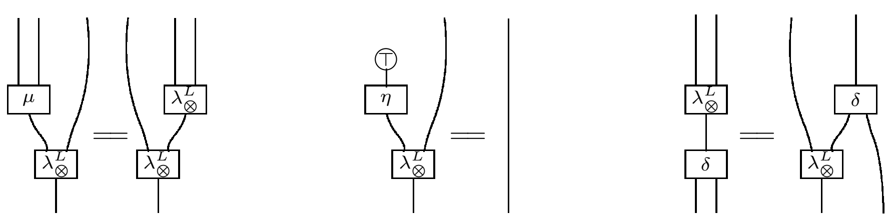
Definition 4.13. (**Linear monads**) A linear monad $\mathbf{T}$ in a linear bicategory $\mathbf{B}$ consists of a compatible monad/comonad pair $\langle\, \langle T, \eta, \mu \rangle, \langle S, \epsilon, \delta \rangle \,\rangle$ on a 0-cell X, together with two actions and two coactions:

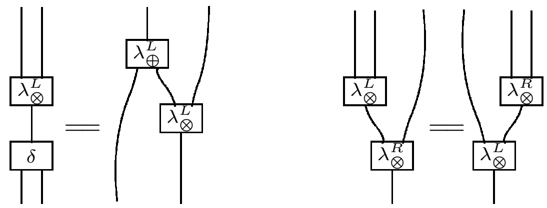
$$\lambda^L_\otimes : T \otimes S \Longrightarrow S \qquad \lambda^R_\otimes : S \otimes T \Longrightarrow S$$
$$\lambda^R_\oplus : T \Longrightarrow T \oplus S \qquad \lambda^L_\oplus : T \Longrightarrow S \oplus T$$

satisfying conditions that say that these are actions (respectively coactions) in the appropriate sense, that μ, δ preserve the actions (respectively coactions), that the actions and coactions are compatible, and that the actions (respectively coactions) are "associative" with respect to each other.

Briefly, these conditions amount to the following. That $\lambda^L_\otimes$ is a (left) action means that $\mu \otimes 1$; $\lambda^L_\otimes = 1 \otimes \lambda^L_\otimes$; $\lambda^L_\otimes$ and η ; $\lambda^L_\otimes = (u^L_\otimes)^{-1}$. There are three other similar pairs of diagrams for the other λ's. That δ preserves the action $\lambda^L_\otimes$ means that $\lambda^L_\otimes$; $\delta = 1 \otimes \delta$; δ_L ; $\lambda^L_\otimes \oplus 1$. There are three other similar diagrams relating δ to $\lambda^R_\otimes$ and μ to the two $\lambda_\oplus$'s. By "compatibility" of $\lambda^L_\otimes$ and $\lambda^L_\oplus$, we intend that $\lambda^L_\oplus \otimes 1$; δ_R ; $1 \oplus \lambda^L_\otimes = \lambda^L_\otimes$; δ; there is a dual diagram for the two λ^L's, and two for the λ^R's. By "associativity" of the two $\lambda_\otimes$'s, we mean that $\lambda^L_\otimes \otimes 1$; $\lambda^R_\otimes = 1 \otimes \lambda^R_\otimes$; $\lambda^L_\otimes$. There is a similar diagram for the $\lambda_\oplus$'s.

A representative set of circuit equations is given below.





But there is another characterization of linear monads, in terms of cyclic linear adjoints. The composition of two cyclic linear adjoints is itself a cyclic linear adjoint. From this we can conclude that a linear monad should also be a cyclic linear adjoint and the unit and counit, multiplication and comultiplication should be (two-way) mates. More explicitly, given a linear monad as above, we construct a cyclic linear adjoint $\langle v, w\rangle : S \Vdash\!\!\dashv T$ with $\tau_v = \lambda_\otimes^L \,;\, \epsilon$, $\gamma_v = \eta \,;\, \lambda_\oplus^L$, $\tau_w = \lambda_\otimes^R \,;\, \epsilon$, $\gamma_w = \eta \,;\, \lambda_\oplus^R$. The necessary commutative diagrams are all straightforward. This suggests the following equivalences.

Proposition 4.14. The following are equivalent.

1 A linear monad.
2 A $\otimes$-monad on a endo-1-cell which has a cyclic linear adjoint such that the unit and multiplication are cyclic.
3 A $\oplus$-comonad on an endo-1-cell which has a cyclic linear adjoint such that the counit and comultiplication are cyclic.

Proof. It remains to show that given a $\otimes$-monad $\langle T, \eta, \mu\rangle$ as specified, with its cyclic adjoint S it determines a linear monad. Let δ, ϵ be the mates of μ, η respectively; then $\lambda_\oplus^L = u_\otimes^L \,;\, \tau\otimes 1 \,;\, \delta_L \,;\, 1\oplus\mu$, and the other λ's are defined dually. It is then straightforward to verify that this does indeed define a linear monad, and moreover, this construction is inverse to the construction of a cyclic linear adjoint from a linear monad. The proof that a linear monad is equivalent to a $\oplus$-comonad with cyclic counit and comultiplication is similar. $\square$

An immediate consequence of this is the following.

Corollary 4.15. A cyclic linear adjoint induces a linear monad on its domain and codomain.

Proposition 4.14 gives an alternative characterization of linear monads, in terms of linear functors, analogous to the standard result that monads (or triples) are monoidal functors from a trivial monoidal category. Let $\mathbf{1}$ be the suspension of the one-object linearly distributive category (so there is one 0-cell $\bullet$, one 1-cell I, one 2-cell, with trivial tensor and par structure). A linear functor $F : \mathbf{1} \longrightarrow \mathbf{B}$, which we shall call a **linear point** as in (Cockett–Seely 1998), is given by a 0-cell X of $\mathbf{B}$ (*viz.* $F(\bullet)$), a pair of 1-cells T, S (*viz.* $F_\otimes(I), F_\oplus(I)$ respectively), and various 2-cells corresponding to the m's needed to make $F_\otimes$ monoidal, the n's needed to make $F_\oplus$ comonoidal, and the ν's needed to make F linear (Definition 2.4). A moment's glance will show that this is exactly the data needed to define a linear monad, and it is easy to show that the coherence conditions in the one definition correspond exactly to those of the other. So we are led to the fact that linear monads correspond to linear points.

Corollary 4.16. Linear monads are the same as linear points, *viz.* linear functors $F\colon \mathbf{1} \longrightarrow \mathbf{B}$.

Proof. The only thing that remains to show is that a linear point is indeed a linear monad. But this is a corollary of the fact that linear functors preserve linear adjoints (Cockett–Seely 1998), and that they preserve the cyclic mate condition, once one notes that $\mathbf{1}$ has a trivial linear monad structure on it. $\square$

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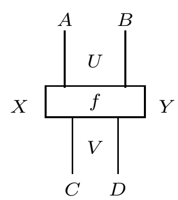
Appendix A. Circuit diagrams

Derivations in the sequent calculus introduced in Section 1 may be represented by graphs (which we call "circuit diagrams"), essentially the version of proof nets which we have used in the previous papers in this series (Cockett–Seely 1992; Cockett–Seely 1997; Blute *et al.* 1996; Blute *et al.* 1996a; Cockett–Seely 1997a; Cockett–Seely 1998). The only change we introduce in this paper is that we may label the areas of the plane in which the graph is located by the types of the logic. (This involves the convention that the "input" wires and the "output" wires extend to the top and bottom respectively of the planar area in which the graph lives; we shall not draw the boundary of this area, leaving it to the reader's imagination.) So, we represent a sequent (labelled f) $A^{X \longrightarrow U}, B^{U \longrightarrow Y} \vdash C^{X \longrightarrow V}, D^{V \longrightarrow Y}$ by the graph



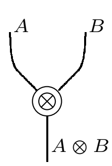
Notice that such sequents are in bijective correspondence with sequents of the form $A \otimes B^{X \longrightarrow Y} \vdash C \oplus D^{X \longrightarrow Y}$, by the sequent calculus rules in Table 1.

If we suppress mention of the typing, such a graph would then be exactly the type of graph used in the previous papers. These are of course equivalent to the proof nets of linear logic, but labelling the planar areas is clearer with this type of circuit diagram.

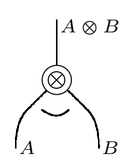
To capture the structure of the sequent calculus, we must introduce appropriate nodes for the connectives and constants. As this follows the earlier "untyped" presentation (Blute *et al.* 1996) very closely, with typing that may be inferred from the sequent rules above, we shall review this informally. Note however that there is a precise syntax that accompanies these graphs, indeed a "term calculus" that permits us to make precise calculations; see (Blute *et al.* 1996) for details. (The typing may be added in a straightforward fashion.)

We have the basic links for the tensor and par, and their units:

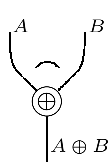
$(\otimes\ I)$



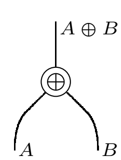
$(\otimes\ E)$

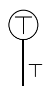


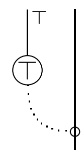
$(\oplus\ I)$

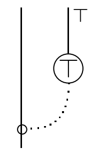


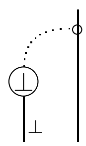
$(\oplus\ E)$

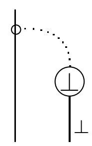


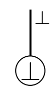
$(\top\ I)$ 

$(\top\ E)^L$ 

$(\top\ E)^R$ 

$(\bot\ I)^L$ 

$(\bot\ I)^R$ 

$(\bot\ E)$ 

We have omitted the typing information. It is clear from these diagrams why the tensor and par units $\top, \bot$ must be endo-2-cells. The $(\otimes\ E)$ and $(\oplus\ I)$ links are "switching" links, as explained in (Blute *et al.* 1996, Section 2.7): in a sense, one should imagine only one of the wires A or B present, but both possibilities must be accounted for. The dotted wires in the $(\top\ E)$ and $(\bot\ I)$ links are called "thinning" links; these are the main tool introduced in (Blute *et al.* 1996) for analyzing the coherence involving the units, and allow us to keep track of where a thinning is introduced in a derivation. Their key property is that they may be moved (in a restricted fashion) without affecting the essential identity of the derivation (or proof circuit). A key proposition of (Blute *et al.* 1996) is the "Rewiring theorem": in its original commutative version, it states that a thinning link may be moved to any wire in its empire. In the noncommutative case, which is the one of interest in the present (planar) context, the analysis of (Blute *et al.* 1996) has been sharpened considerably by Robert R. Schneck (Schneck 1998).

A.1. *Circuit equivalences*

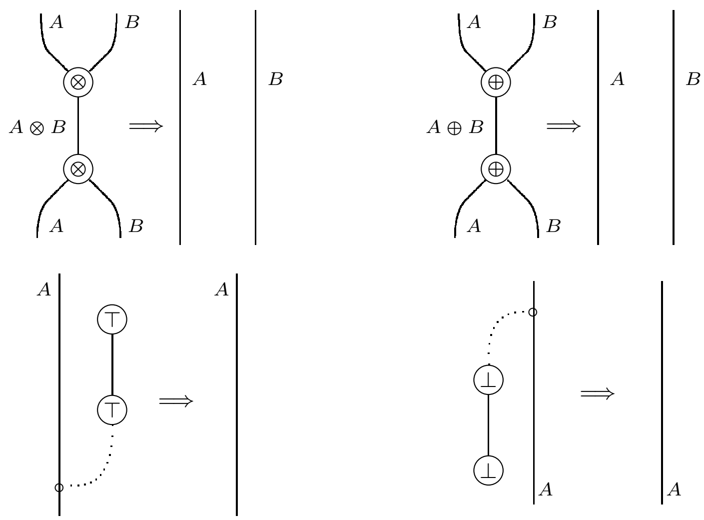
We intend that these circuit diagrams represent 2-cells in a suitable bicategory, and so it is evident that some equivalence relation will be required. Essentially, this will be generated by graph rewrites as in the previous papers (Cockett–Seely 1992; Cockett–Seely 1997; Blute *et al.* 1996; Blute *et al.* 1996a; Cockett–Seely 1997a; Cockett–Seely 1998)—we shall continue to follow the presentation of (Blute *et al.* 1996), and in particular, for the more formal syntax of the term calculus, the reader should refer to that paper.

We list the reduction and expansion rewrites in Table 2 (again, we omit the typing), and a representative sample of the "rewiring" rewrites is given in Table 3 (the remaining ones may be obtained by duality—a full listing is given in (Blute *et al.* 1996)). The numbers labelling the equations refer to the numbering of the rewirings in (Blute *et al.* 1996), and are repeated here for ease of reference.

The point of this is that the circuit diagrams fully and faithfully capture the categorical structure. Analogous to the results in (Blute *et al.* 1996), we may claim that for a generating 2-graph and equations, the bicategory of circuits generated by the 2-graph (considered as generating components and equivalences) is the free linear bicategory generated by the 2-graph and equations. The proof of this claim is essentially the one given in (Blute *et al.* 1996).

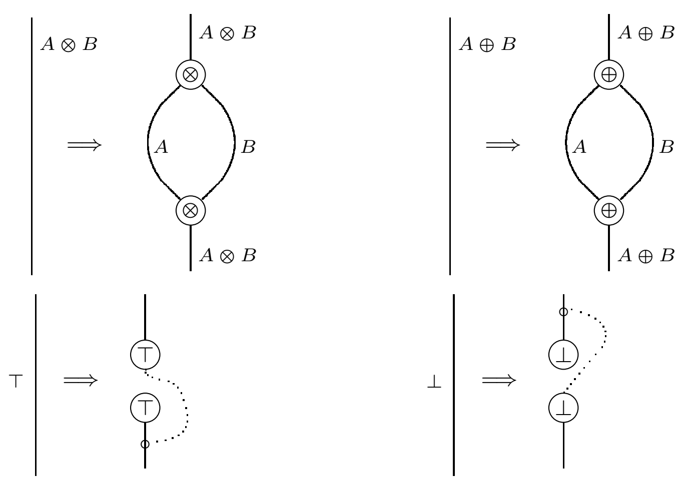
Table 2. *Reduction/Expansion rewrites*

Reductions



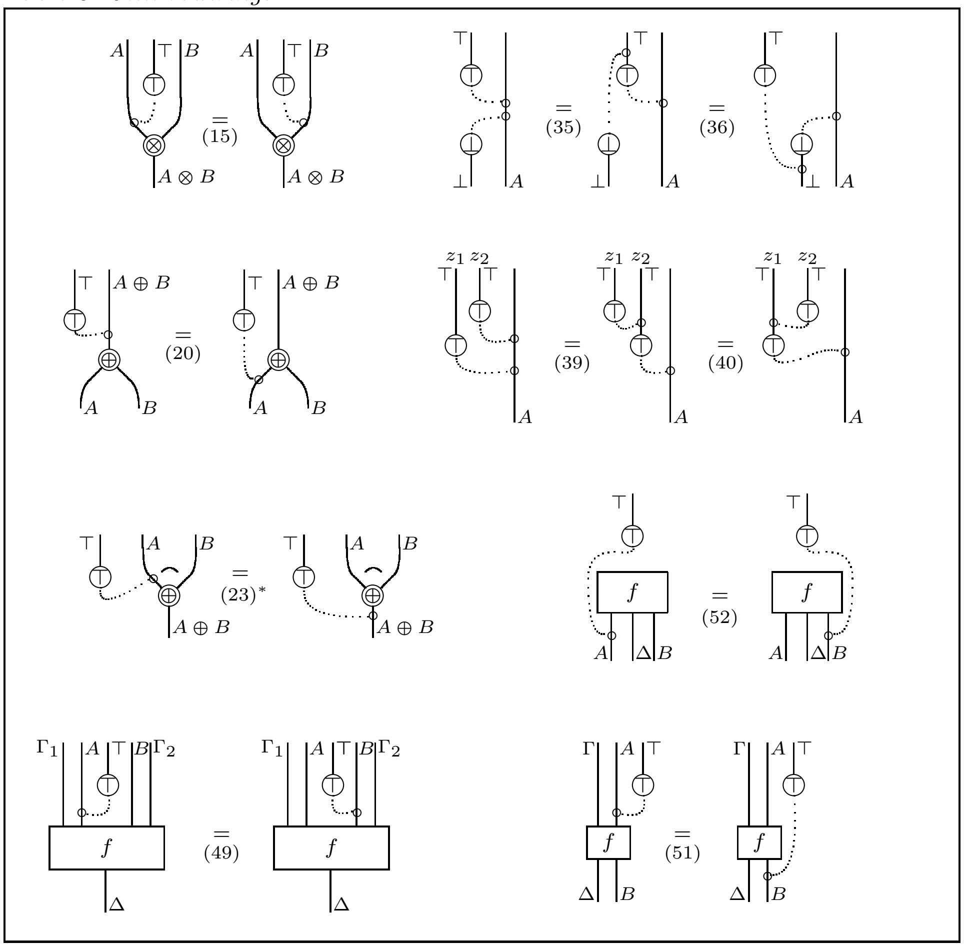
(and the two evident dual rewrites.)

Expansions



(and the two evident dual rewrites.)

Table 3. *Unit rewirings*



It is perhaps worth mentioning, however, that this rewrite system is not confluent on planar circuits because of the restrictions on rewiring moves on the two units. In (Blute *et al.* 1996) we avoided this problem by allowing the unit reductions as equations, which allows one to introduce "floating unit barbells" which enable one to simulate moves with one type of unit using a barbell of the other type of unit. As we pointed out there, however, this has the undesirable consequence that arbitrarily many such barbells may be introduced into a circuit, complicating the decision problem considerably. In (Schneck 1998) an improvement on this situation is made by introducing new links and new rewrites; in essence what Schneck does is introduce *negation* links for the two units (since each unit is the complement of the other, in the evident way (Cockett–Seely 1997a)) with the appropriate reductions for these (treated as equations). With this, the two units gain equivalent mobility. Schneck shows that any net containing these negation links is equivalent to one without them, in a way preserving the circuit equivalence (of course, the unit reductions must then be bidirectional). He then can improve upon the coherence results of (Blute *et al.* 1996), in particular, obtaining a decision procedure for any theory for which all non-logical axioms ("components" in the terminology of (Blute *et al.* 1996)) have no nontrivial unit inputs or outputs. Again, the reader is referred to (Schneck 1998) for the details.