Type (a) (ii). $(t \times Y)^* \underset{\Delta_Y}{\Sigma} \varphi \to \underset{\langle X, t\rangle}{\Sigma} t^*\varphi$: This is represented by the derivation (II) on p. 517.

Type (b). $(B \times t)^* \underset{r \times Y}{\Sigma} \varphi \to \underset{r \times X}{\Sigma} (A \times t)^* \varphi$: This is represented by the following derivation (III):

(III)

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\langle ra, y\rangle = \langle b, tx\rangle \wedge \varphi(a, y)}{\langle ra, y\rangle = \langle b, tx\rangle}\,(\wedge\,\mathrm{E})\,(1)}{ra = b}\,(\mathrm{pE}) \qquad \dfrac{\top}{x = x}\,(\mathrm{R})}{\langle ra, x\rangle = \langle b, x\rangle}\,(\mathrm{pI}) \qquad \dfrac{\dfrac{\dfrac{(1)}{\langle ra, y\rangle = \langle b, tx\rangle}\,(\wedge\,\mathrm{E})}{y = tx}\,(\mathrm{pE}) \qquad \dfrac{(1)}{\varphi(a, y)}\,(\wedge\,\mathrm{E})}{\varphi(a, tx)}\,(\mathrm{sub})}{\langle ra, x\rangle = \langle b, x\rangle \wedge \varphi(a, tx)}\,(\wedge\,\mathrm{I})}{\exists\langle \alpha, \eta\rangle\,(\langle r\alpha, \eta\rangle = \langle b, tx\rangle \wedge \varphi(\alpha, \eta))}\,(\exists\,\mathrm{I}) \qquad \exists\langle \alpha, \xi\rangle\,(\langle r\alpha, \xi\rangle = \langle b, x\rangle \wedge \varphi(\alpha, t\xi))}{\exists\langle \alpha, \xi\rangle\,(\langle r\alpha, \xi\rangle = \langle b, x\rangle \wedge \varphi(\alpha, t\xi))}\,(\exists\,\mathrm{E})\,(1)$$

(The "dual" case $(r \times Y)^* \underset{B \times t}{\Sigma} \varphi \to \underset{A \times t}{\Sigma} (r \times X)^* \varphi$ follows immediately from this, under evident substitutions.)

It must be checked that these are inverse to the morphisms $\Sigma_r t^*\varphi \to t'^*\Sigma_s\varphi$ (for suitable $t, t', r, s$) – this amounts to writing out the various compositions and reducing them to the identity derivations. The details are tedious: we give one example:

$$\Delta_Y^* \underset{t \times Y}{\Sigma} \varphi \to \Sigma_t\langle X, t\rangle^* \varphi \to \Delta_Y^* \underset{t \times Y}{\Sigma} \varphi = \mathrm{id}.$$

We must show that the derivation (IV) on p. 519 is equivalent to (id). (We will suppose $\varphi$ atomic, so that the use of (sub) is correct.) It is not difficult to reduce this to its normal form (V) on p. 520 ((∃ Perm), (∃ Red), and (∧ Red)). Using (T Coh), (= Exp), and (S Coh), this is equivalent to (VI) on p. 520. This in turn (using (= Exp)) is equivalent to (VII) on p. 521. Using (= Simp), (= Red), and (S Coh), we arrive at the equivalent derivation (VIII) on p. 521, and this, using (∧ Exp) and (∃ Exp), is easily shown to be equivalent to (id).

The other parts of the proof are similar: moreover, we shall have more to say about this in § 8 so we will not give further details here. So this completes the proof of the theorem.

## § 5. Construction 2: Hyperdoctrine → LPCE

For any hyperdoctrine $\boldsymbol{P}$ (over $\boldsymbol{T}$) there is a corresponding theory $T_{\boldsymbol{P}}$ in LPCE. Briefly, the theory $T := T_{\boldsymbol{P}}$ is constructed as follows: the sorts of the language $\mathscr{L}$ of $T$ are the objects of $\boldsymbol{T}$, the function symbols are morphisms of $\boldsymbol{T}$, predicate symbols are objects of $\boldsymbol{P}$ (sorted by fibres) and rules of inference the morphisms of $\boldsymbol{P}$. If $gf = h$, then we add a reduction $P_f \underset{P_g}{\Rightarrow} P_h$, where $P_i$ is the rule given by $i$.



(IV)

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\langle tx, y'\rangle = \langle y, y\rangle \wedge \varphi(x, y')}{\langle tx, y'\rangle = \langle y, y\rangle}\,(\wedge\,\mathrm{E})\,(1)}{\langle tx, y'\rangle = \langle y, y\rangle}\,(\mathrm{pE}) \qquad \dfrac{(1)}{tx = y}\,(\mathrm{S})}{\langle y, y\rangle = \langle tx, y'\rangle}\,(\wedge\,\mathrm{E}) \qquad \dfrac{(1)}{y' = y \quad y = tx}\,(\mathrm{T})}{\langle y, y\rangle = \langle tx, y'\rangle}\,(\wedge\,\mathrm{E}) \qquad \dfrac{(1)}{y' = tx\ \varphi(x, y')}\,(\mathrm{sub})}{tx = y}\,(\mathrm{pE}) \qquad \dfrac{\varphi(x, tx)}{}}{tx = y \wedge \varphi(x, tx)}\,(\wedge\,\mathrm{I})}{\exists\langle \xi, \eta\rangle\,(\langle t\xi, \eta\rangle = \langle y, y\rangle \wedge \varphi(\xi, \eta))}\,(\exists\,\mathrm{I}) \qquad \exists\langle \xi, \eta\rangle\,(\langle t\xi, \eta\rangle = \langle y, y\rangle \wedge \varphi(\xi, \eta))}{\exists\langle \xi, \eta\rangle\,(\langle t\xi, \eta\rangle = \langle y, y\rangle \wedge \varphi(\xi, \eta))}\,(\exists\,\mathrm{E})\,(1) \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{(2)}{tx = y \wedge \varphi(x, tx)}}{tx = y}\,(\wedge\,\mathrm{E}) \qquad \dfrac{(2)}{tx = y}\,(\wedge\,\mathrm{E})}{\langle tx, tx\rangle = \langle y, y\rangle\ \varphi(x, tx)}\,(\wedge\,\mathrm{I})}{\langle tx, tx\rangle = \langle y, y\rangle \wedge \varphi(x, tx)}\,(\wedge\,\mathrm{I})}{\exists\langle \xi, \eta\rangle\,(\langle t\xi, \eta\rangle = \langle y, y\rangle \wedge \varphi(\xi, \eta))}\,(\exists\,\mathrm{I}) \qquad \exists\langle \xi, \eta\rangle\,(\langle t\xi, \eta\rangle = \langle y, y\rangle \wedge \varphi(\xi, \eta))}{\exists\langle \xi, \eta\rangle\,(\langle t\xi, \eta\rangle = \langle y, y\rangle \wedge \varphi(\xi, \eta))}\,(\exists\,\mathrm{E})\,(2)}{\exists\langle \xi, \eta\rangle\,(\langle t\xi, \eta\rangle = \langle y, y\rangle \wedge \varphi(\xi, \eta))}\,\exists\mathrm{E}$$