Given $\varphi(y) \xrightarrow{\bar{P}} \forall\xi(t\xi = y \supset \gamma(\xi))$ over Y, with P a derivation in $\bar{P}$, we obtain $\varphi(t(x)) \to \gamma(x)$, represented by the derivation:

(ii)

$$\begin{array}{c} \varphi(t(x)) \\ P(t) \\ \dfrac{(\forall\,E)\ \dfrac{\forall\xi(t\xi = t(x) \supset \gamma(\xi))}{t(x) = t(x) \supset \gamma(x)} \qquad \dfrac{\top}{t(x) = t(x)}\ (R)}{\gamma(x)}\ (\supset E) \end{array}$$

Conversely, given $\varphi(t(x)) \xrightarrow{\bar{P}} \gamma(x)$ over X, and P in $\bar{P}$, we define a proof

$$\varphi(y) \to \forall\xi(t\xi = y \supset \gamma(\xi))$$

represented by:

(iii)

$$\dfrac{\dfrac{\begin{array}{c}\dfrac{(\mathrm{S})\ \dfrac{\overset{(1)}{t(x) = y}}{y = t(x)} \quad \varphi(y)}{\varphi(t(x))}\ (\mathrm{sub})' \\ P \\ \gamma(x)\end{array}}{t(x) = y \supset \gamma(x)}\ (\supset I)\ (1)}{\forall\xi(t\xi = y \supset \gamma(\xi))}\ (\forall\,I)$$

((sub)′ is the suitable derived rule.) It must be checked that these processes are in fact well-defined with respect to the equivalence relation, but that this is so should be clear from the forms of the above derivations. (This remark will also hold when we come to consider the Beck conditions below.)

Also, we must check that these correspondences do in fact determine an adjunction. Perhaps the simplest way is to verify the triangle equalities for the induced natural transformations $\eta_\Sigma\colon I \to t^*\Sigma_t$, $\varepsilon_\Sigma\colon \Sigma_t t^* \to I$, $\eta_\Pi\colon I \to \Pi_t t^*$, $\varepsilon_\Pi\colon t^*\Pi_t \to I$, checking also that these are in fact natural transformations. This involves writing out various derivations in full, and collapsing and expanding them according to the operations given above: we sketch one part of the proof as an example: namely, that the triangle equality $\Pi_t\varepsilon_\varphi\eta_{\Pi_t\varphi} = \mathrm{id}_{\Pi_t\varphi}$ holds for the adjunction $t^* \dashv \Pi_t$, for φ over X. We shall not define ε, η here – the definitions are immediate from the correspondences (ii), (iii) above – but when they are written out in full, and when the definition of $\Pi_t\bar{P}$, for a proof $\bar{P}$, is also written in full, it would be seen that the composition $\Pi_t\varepsilon_\varphi \cdot \eta_{\Pi_t\varphi}$ is represented by the following derivation (perhaps it should be mentioned that the subderivation above the topmost occurrence of $\forall\xi(t\xi = tx \supset \varphi(\xi))$ is the expanded form of

$$\dfrac{\dfrac{tx = y}{y = tx}\ (\mathrm{S}) \qquad \forall\xi(t\xi = y \supset \varphi(\xi))}{\forall\xi(t\xi = tx \supset \varphi(\xi))}\ (\mathrm{sub})'$$

(a derived rule), that one might expect from the correspondence (iii) above; recall that (T) is a special case of (= E) or (sub)):

$$\dfrac{\dfrac{\forall\xi(t\xi = y \supset \varphi(\xi))}{tx' = y \supset \varphi(x')}\ (\forall\,E) \qquad \dfrac{\overset{(1)}{tx' = tx} \quad \overset{(2)}{tx = y}}{tx' = y}\ (\mathrm{T})}{\varphi(x')}\ (\supset E)$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\varphi(x')}{tx' = tx \supset \varphi(x')}\ (\supset I)\ (1)}{\forall\xi(t\xi = tx \supset \varphi(\xi))}\ (\forall\,I)}{tx = y \supset \forall\xi(t\xi = tx \supset \varphi(\xi))}\ (\supset I)\ (2)}{\forall\xi'(t\xi' = y \supset \forall\xi(t\xi = t\xi' \supset \varphi(\xi)))}\ (\forall\,I)$$

$$\dfrac{\dfrac{\forall\xi'(t\xi' = y \supset \forall\xi(t\xi = t\xi' \supset \varphi(\xi)))}{tx = y \supset \forall\xi(t\xi = tx \supset \varphi(\xi))}\ (\forall\,E)\ (3) \qquad \overset{(3)}{tx = y}}{\forall\xi(t\xi = tx \supset \varphi(\xi))}\ (\supset E)$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\forall\xi(t\xi = tx \supset \varphi(\xi))}{tx = tx \supset \varphi(x)}\ (\forall\,E) \qquad \dfrac{\top}{tx = tx}\ (=I)}{\varphi(x)}\ (\supset E)}{tx = y \supset \varphi(x)}\ (\supset I)\ (3)}{\forall\xi(t\xi = y \supset \varphi(\xi))}\ (\forall\,I)$$

We must show this is equivalent to the identity derivation. Straightforward uses of ($\forall$ Red) and ($\supset$ Red) give us the normal form of the above:

$$\dfrac{\dfrac{\dfrac{\dfrac{\forall\xi(t\xi = y \supset \varphi(\xi))}{tx = y \supset \varphi(x)} \qquad \dfrac{\dfrac{\top}{tx = tx} \quad tx = y}{tx = y}}{\varphi(x)}}{tx = y \supset \varphi(x)}}{\forall\xi(t\xi = y \supset \varphi(\xi))}$$

Finally, (= Exp), ($\supset$ Exp), and ($\forall$ Exp) show this is equivalent to (id) as required.

The other parts of the proof that we have defined an adjunction proceed much like this, so we shall omit the rest of the details.

(5′) (ii) As we remarked earlier, there are only three cases we need consider. (For a more general theory T, there may be others, but then the axioms and rules of T that give rise to further pullbacks will permit a similar proof to go through.) In fact (as the categorist will have suspected from adjointness considerations) one direction