Also, given a pullback

(D)
$$\begin{array}{ccc} X & \xrightarrow{t} & Y \\ {\scriptstyle r}\downarrow & & \downarrow{\scriptstyle s} \\ X' & \xrightarrow{t'} & Y' \end{array}$$

then for any object Z,

(i)$_D$
$$\begin{array}{ccc} X \times Z & \xrightarrow{t \times Z} & Y \times Z \\ {\scriptstyle r \times Z}\downarrow & & \downarrow{\scriptstyle s \times Z} \\ X' \times Z & \xrightarrow{t' \times Z} & Y' \times Z \end{array}$$

("Preservation under products") is a pullback. Finally, if

(D′)
$$\begin{array}{ccc} Y & \xrightarrow{p} & Z \\ {\scriptstyle s}\downarrow & & \downarrow{\scriptstyle q} \\ Y' & \xrightarrow{p'} & Z' \end{array}$$

is a pullback, then so is

(ii)$_{DD'}$
$$\begin{array}{ccc} X & \xrightarrow{pt} & Z \\ {\scriptstyle r}\downarrow & & \downarrow{\scriptstyle q} \\ X' & \xrightarrow{p't'} & Z' \end{array}$$

("Preservation under composition").

If we are only interested in the pure logic with equality (and not in any other rules a theory in LPCE might introduce), it will suffice if we suppose the Beck condition only for the cases (a), (b), (c), and (d), and that it is preserved under the pullback formation rules (i), (ii), given above. In fact, some of this is automatic: Beck must hold in case (d), since for any isomorphism s, $(s^{-1})^* = \Sigma_s$; also Beck must be preserved under (ii). (In fact, on most of those occasions we actually use the Beck condition, it will be the case that it is preserved under (i) also, (e.g., when the $\varphi \in |\boldsymbol{P}(Y \times Z)|$ is of the form $\pi_Y^* \psi$, for $\psi \in |\boldsymbol{P}(Y)|$).) So we need only assert Beck for cases (a), (b), and (c), and its preservation under (i). We shall return to this point in § 8.

§ 4. Construction 1: LPCE → Hyperdoctrine

For any theory T expressed in LPCE, there is a corresponding hyperdoctrine $\boldsymbol{P}_T$ (over $\boldsymbol{T}_T$). We'll construct $\boldsymbol{P}_T := \boldsymbol{P}_0$ for the case of the empty theory T_0 (so, insofar as $\boldsymbol{P}_T$ "is" T, the resulting $\boldsymbol{P}_0$ "is" LPCE; the more general construction will then

(I hope) be clear). Objects of $\boldsymbol{T}_0$ are the types of the language $\mathscr{L}$. Morphisms of $\boldsymbol{T}_0$ are essentially the terms. (Some technical fiddling is necessary here so that $\boldsymbol{T}_0$ is a category with finite products, but this is straightforward. For details see SEELY [18]. For any type X, the objects of the fibre over X are formulae whose free variables are of type X. Morphisms of the fibre are proofs (i.e. equivalence classes of derivations) – again the domains and codomains are obvious. (For arbitrary T, we can expect to get more morphisms in the fibres; this will be the only change. Of course, changing $\mathscr{L}$ will affect $\boldsymbol{T}$ and $|\boldsymbol{P}|$ also.)

Theorem. *$\boldsymbol{P}_0$ as constructed above is a hyperdoctrine.*

Proof. (0) $\boldsymbol{T}_0$ has by construction all finite products, including the empty product 1.

(1) For a term $t: X \to Y$, we must construct $t^*: \boldsymbol{P}_0(Y) \to \boldsymbol{P}_0(X)$: this is just "substitute t for y", i.e. $t^*\varphi$ is $\varphi(t)$.

(2) The cartesian closed structure is given using $\wedge$ for products and $\supset$ for exponentiation. The deduction rules and definition of equivalence account for the proper structure; this is more or less proven in SEELY [19].

(3) Similarly, $\vee$ for coproduct and $\perp_X$ for 0_X give the required structure.

(4′) This is trivial by definition of substitution:

$$\varphi(t) \supset \varphi'(t) = (\varphi \supset \varphi')\,(t).$$

(5′) (i) For $t: X \to y$, φ over X, we define

$$\Sigma_t\varphi =_{df} \exists\xi(t\xi = y \wedge \varphi(\xi)), \qquad \Pi_t\varphi =_{df} \forall\xi(t\xi = y \supset \varphi(\xi)).$$

Σ_t, Π_t are defined on proofs as follows: suppose $\gamma \xrightarrow{\bar{P}} \varphi$ is a proof over X, represented by a derivation P. Then $\Sigma_t\bar{P}$ is the proof represented by the derivation:

(i)
$$\dfrac{\exists\xi(t\xi = y \wedge \gamma(\xi)) \qquad \dfrac{\dfrac{\dfrac{\dfrac{tx = y \wedge \gamma(x)}{[\gamma(x)]}\,(\wedge\text{E}) \atop P}{\varphi(x)} \qquad \dfrac{tx = y \wedge \gamma(x)}{tx = y}\,(\wedge\text{E})}{tx = y \wedge \varphi(x)}\,(\wedge\text{I})}{\exists\xi(t\xi = y \wedge \varphi(\xi))}\,(\exists\text{I})}{\exists\xi(t\xi = y \wedge \varphi(\xi))}\,(\exists\text{E})$$

$\Pi_t\bar{P}$ is defined similarly.

These are the required adjoints: we must show, for γ over X, φ over Y, bijections

$$\frac{\Sigma_t\gamma \to \varphi \quad (\text{over } Y)}{\gamma \to t^*\varphi \quad (\text{over } X)} \qquad \frac{\varphi \to \Pi_t\gamma \quad (\text{over } Y)}{t^*\varphi \to \gamma \quad (\text{over } X)}.$$

We do this for Π, the proof for Σ is similar.