a subtree of P_i by the result of applying an operation to the subtree, or P_i is obtained from P_{i+1} this way.

Three equivalence schema are useful:

Lemma. *If $Q_i(t, t')$: $t = t' \vdash \varphi(t, t')$ $(i = 0, 1)$ are derivations such that $Q_0(t, t) \equiv$ $\equiv Q_1(t, t)$, for any t, then $Q_0(t, t') \equiv Q_1(t, t')$, for any t, t'.*

Corollary 1. *For any derivations $Q_i(t, t')$: $\Gamma \vdash t = t'$ $(i = 0, 1)$, $Q_0(t, t') \equiv Q_1(t, t')$.*

Corollary 2. *For any derivations P, P_1, P_2, P_3 there are equivalences:*

(i) (S Coh): $$\text{(S)}\,\cfrac{\text{(S)}\,\cfrac{\overset{P_1}{t = t'}}{t' = t}}{t = t'} \equiv \overset{P_1}{t = t'}$$

(ii) (T Coh): $$\cfrac{\text{(T)}\,\cfrac{\overset{P_1}{t = t'}\quad \overset{P_2}{t' = t''}}{t = t''}\qquad t'' = t'''}{t = t'''} \equiv \cfrac{P_1\ \cfrac{\overset{P_2}{t' = t''}\quad \overset{P_3}{t'' = t'''}}{t = t'\quad t' = t'''}\,\text{(T)}}{t = t'''}\,\text{(T)}$$

(iii) (= Simp): *If (φ is atomic and) t, t' appear in a dummy position, (so P is a derivation of $\varphi(t')$)*

$$\cfrac{\overset{P_1}{t = t'}\quad \overset{P}{\varphi(t)}}{\varphi(t')} \equiv \overset{P}{\varphi(t')}.$$

Proofs. The Lemma is an immediate consequence of (= Exp). Then Corollary 1 follows, since $t = t$ is a "terminal object": there is, up to equivalence, exactly one derivation $\Gamma \vdash t = t$, for any Γ. This can be shown directly, via (! Red) and (R Coh), or by using the methods and results of § 4.

Similarly, Corollary 2(i) and (ii) are immediate. To prove (= Simp), replace t' by t and apply (= Red) to see

$$\cfrac{\overset{P_1(t,t)}{t = t}\quad \overset{P}{\varphi(t)}}{\varphi(t)} \equiv \overset{P}{\varphi(t)}. \ \square$$

§ 3. Hyperdoctrines

Definition. A $\boldsymbol{T}$-category $\boldsymbol{P}$ is a *hyperdoctrine* iff

(0) $\boldsymbol{T}$ has finite products and terminal object 1,
(1) $\boldsymbol{P}$ is an indexed category over $\boldsymbol{T}$ ("a $\boldsymbol{T}$-category"),
(2) for each object X of $\boldsymbol{T}$, the fibre $\boldsymbol{P}(X)$ is cartesian closed, and furthermore,
(3) has finite coproducts and an initial object 0_X,
(4) for each morphism t of $\boldsymbol{T}$, the "inverse image" functor t^* preserves the structure of (2), (3),
(5) $\boldsymbol{P}$ has $\boldsymbol{T}$-sums and $\boldsymbol{T}$-products.

As this definition is non-standard, (it differs from that given in LAWVERE [10]) perhaps we should elaborate. This will also allow us to fix notation. The definition of an indexed category is standard enough, (although we only suppose finite products in the base category, and not arbitrary finite limits). (See PARÉ-SCHUMACHER [13].) The point is this: an indexed category is equivalent to a (pseudo)-functor $\boldsymbol{T}^{\mathrm{op}} \to \mathscr{C}at$. Hence the notation $\boldsymbol{P}(X)$ for the "fibre" over X. The "inverse image" functor t^*, for $t\colon X \to Y$ a morphism of $\boldsymbol{T}$, is then given as $t^* = \boldsymbol{P}(t)\colon \boldsymbol{P}(Y) \to \boldsymbol{P}(X)$. Clauses (2)–(4) are straightforward. Clause (5) can be replaced by:

(5′) (i) for each $t\colon X \to Y$ in $\boldsymbol{T}$, t^* has adjoints

$$\Sigma_t \dashv t^* \dashv \Pi_t,$$

and

(5′) (ii) if
$$\begin{tikzcd} X \arrow[r, "t"] \arrow[d, "r"'] & Y \arrow[d, "s"] \\ X' \arrow[r, "t'"] & Y' \end{tikzcd}$$
is a pullback in $\boldsymbol{T}$, and $\varphi \in |\boldsymbol{P}(Y)|$,

then the morphism $\Sigma_r t^* \varphi \to t'^* \Sigma_s \varphi$ is an isomorphism.

Note that then we can replace

(4) by:

(4′) the "inverse image" functors t^* preserve exponentiation.

(All the rest of the structure of (2) and (3) must be preserved, because of the existence of both adjoints.) This is equivalent to

(4″) for each $t\colon X \to Y$ in $\boldsymbol{T}$, $\varphi \in |\boldsymbol{P}(X)|$, $\psi \in |\boldsymbol{P}(Y)|$, the morphism

$$\Sigma_t(t^*\psi \wedge \varphi) \to \psi \wedge \Sigma_t \varphi$$

is an isomorphism.

Condition (4′) (or (4″)) is called *Frobenius Reciprocity*; condition (5′) (ii) is called *Beck condition*. Note that the analogous condition for Π must hold; just consider adjoints.

Some remarks on pullbacks and the Beck condition: Any category with finite products must have the following types of pullbacks:

(a)
$$\begin{tikzcd} X \arrow[r, "\langle X, t\rangle"] \arrow[d, "t"'] & X \times Y \arrow[d, "t \times X"] \\ Y \arrow[r, "\Delta_Y"] & Y \times Y \end{tikzcd}$$

(b)
$$\begin{tikzcd} A \times X \arrow[r, "A \times t"] \arrow[d, "r \times X"'] & A \times Y \arrow[d, "r \times Y"] \\ B \times X \arrow[r, "B \times t"] & B \times Y \end{tikzcd}$$

(c)
$$\begin{tikzcd} X \arrow[r, "X"] \arrow[d, "X"'] & X \arrow[d, "\Delta_X"] \\ X \arrow[r, "\Delta_X"] & X \times X \end{tikzcd}$$

(d)
$$\begin{tikzcd} A \times X \arrow[r, "r \times t"] \arrow[d, "s \wr"'] & B \times Y \arrow[d, "\wr s"] \\ X \times A \arrow[r, "t \times r"] & Y \times B \end{tikzcd}$$

where s is the "switch coordinates" isomorphism.