2. Operations on derivations

As in SEELY [19], we will be interested in a number of operations on derivations: reductions, expansions, and permutations. Most of these are standard, and can be found in PRAWITZ [14], [15]. In addition, we have these new rules:

(! Red) $\begin{array}{c}\varphi\\ P\\ \top_X\end{array} \Rightarrow \dfrac{\varphi}{\top_X}\,(!)$ (φ the sole assumption of P)

($\perp$ Red) $\begin{array}{c}\perp_X\\ P\\ \varphi\end{array} \Rightarrow \dfrac{\perp_X}{\varphi}\,(\perp)$ ($\perp_X$ the sole assumption of P)

(Strictly, (! Red) and ($\perp$ Red) apply only when φ is atomic and different from $\top_X$ (respectively $\perp_X$). If φ is $\top_X$ (respectively $\perp_X$) these reductions will be understood to be to the identity derivation. Also, if φ is not atomic, the reductions will be to the appropriate derived rule.)

(= Red) $$\dfrac{\dfrac{\top}{t = t}\quad \begin{array}{c}P\\ \varphi(t)\end{array}}{\varphi(t)} \Rightarrow \begin{array}{c}P\\ \varphi(t)\end{array} \quad (\text{atomic } \varphi)$$

(= Exp) $$\begin{array}{c}P\\ t = t'\\ Q(t,t')\\ \varphi(t,t')\end{array} \Rightarrow \dfrac{\begin{array}{c}P\\ t = t'\end{array}\quad \begin{array}{c}\dfrac{\top}{t = t}\\ Q(t,t)\\ \varphi(t,t)\end{array}}{\varphi(t,t')}$$

(provided the RHS is a derivation)

($\vee$ Perm) $$\begin{array}{c}\dfrac{\begin{array}{c}P\\ \varphi\vee\psi\end{array}\quad \begin{array}{c}[\varphi]\\ Q\\ \theta\end{array}\quad \begin{array}{c}[\psi]\\ R\\ \theta\end{array}}{[\theta]}\\ S\\ \gamma\end{array} \Rightarrow \dfrac{\begin{array}{c}P\\ \varphi\vee\psi\end{array}\quad \begin{array}{c}[\varphi]\\ Q\\ [\theta]\\ S\\ \gamma\end{array}\quad \begin{array}{c}[\psi]\\ R\\ [\theta]\\ S\\ \gamma\end{array}}{\gamma}$$

($\exists$ Perm) $$\begin{array}{c}\dfrac{\begin{array}{c}P\\ \exists\xi\varphi(\xi)\end{array}\quad \begin{array}{c}[\varphi(x)]\\ Q\\ \theta\end{array}}{[\theta]}\\ S\\ \gamma\end{array} \Rightarrow \dfrac{\begin{array}{c}P\\ \exists\xi\varphi(\xi)\end{array}\quad \begin{array}{c}[\varphi(x)]\\ Q\\ [\theta]\\ S\\ \gamma\end{array}}{\gamma}$$

And a "coherence" rule for equality:

(R Coh) $$\dfrac{\dfrac{\begin{array}{c}P\\ t = t\end{array}}{\top}}{t = t} \Rightarrow \begin{array}{c}P\\ t = t\end{array}$$

Remarks. (1) As shown in SEELY [19] these permutation rules amount to streng thening PRAWITZ' $\vee$ expansion and $\exists$ expansion to:

($\vee$ Exp) $$\begin{array}{c}P_0\\ [\varphi\vee\psi]\\ P_1\\ \theta\end{array} \Rightarrow \dfrac{\begin{array}{c}P_0\\ \varphi\vee\psi\end{array}\quad \begin{array}{c}\dfrac{[\varphi]}{\varphi\vee\psi}\\ P_1\\ \theta\end{array}\quad \begin{array}{c}\dfrac{[\psi]}{\varphi\vee\psi}\\ P_1\\ \theta\end{array}}{\theta}$$

(provided the RHS is a derivation)

($\exists$ Exp) $$\begin{array}{c}P_0\\ [\exists\xi\varphi(\xi)]\\ P_1\\ \theta\end{array} \Rightarrow \dfrac{\begin{array}{c}P_0\\ \exists\xi\varphi(\xi)\end{array}\quad \begin{array}{c}\dfrac{[\varphi(x)]}{[\exists\xi\varphi(\xi)]}\\ P_1\\ \theta\end{array}}{\theta}$$

(provided the RHS is a derivation).
(There is a (= Perm) in the same spirit, but we won't need to refer to it.)

(2) There are some derived rules of interest:

In (= E), some of the terms s' may be identical to the corresponding s, and, adop ing the convention that $\top$ may be dropped as an assumption whenever it occurs, too can assumptions of the form $s = s$.

Some special cases of (= E) should be noted: for any terms t, t', t'', typed $X \to$ say, the following are (derived) rules:

(S) $$\dfrac{t = t'}{t' = t} \qquad \left(\text{viz. } \dfrac{t = t' \quad \dfrac{\top}{t = t} \quad \dfrac{\top}{t = t}\,(= \mathrm{I})}{t' = t}\,(= \mathrm{E})\right)$$

(T) $$\dfrac{t = t' \quad t' = t''}{t = t''} \qquad \left(\text{viz. } \dfrac{\dfrac{\top}{t = t}\,(= \mathrm{I}) \quad t' = t'' \quad t = t'}{t = t''}\,(= \mathrm{E})\right)$$

(In keeping with these notations, we shall also use (R) for (= I) and (sub) for (= E

(3) We will be casual in using the meta-notation for types and "terms of a giv type". For example, (= E) will frequently appear as

$$\dfrac{t = t' \quad \varphi(t)}{\varphi(t')}.$$

(4) A given theory in LPCE may impose further operations on derivations, in a dition to non logical rules. We regard a theory as given by its language, non-logic deduction rules, and operations on derivations.

We define an equivalence relation $\equiv$ on derivations in the natural way: $\equiv$ is t smallest equivalence relation making all of the given operations equivalences. E plicitly, derivations P, P' are equivalent iff there are derivations $P = P_1, P_2, \ldots$ $P_k = P'$ ($k \geq 1$) so that for each $i < k$, either P_{i+1} is obtained from P_i by replaci