the "stratification" of formulae and derivations. So we suppose our language $\mathscr{L}$ to contain:

sort symbols: $X, Y, Z, \ldots$

free variables of each sort: $x, x', \ldots, y, y', \ldots, z, \ldots$

bound variables of each sort: $\xi, \xi', \ldots, \eta, \eta', \ldots, \zeta, \ldots$

sorted function symbols: $f, g, \ldots$

sorted predicate symbols: $P, Q, \ldots$

logical symbols: $\top, \bot, \wedge, \vee, \supset, \exists, \forall$.

(Among the predicate symbols we assume a binary predicate E_X for each sort X, which will be interpreted as equality for that sort.)

To be able to treat functions and predicates as if they were unary, and to simplify sorting terms and formulae, we introduce the meta-mathematical notion of "type": a type is a finite sequence (here written as a product) of sorts. So if X, Y are sorts, then $X \times Y$ is a type. The empty sequence of sorts is denoted 1. "A free variable of type $X \times Y$" is understood to mean a sequence $\langle x, y\rangle$ of free variables of sorts X, Y respectively, and similarly for other terms. The obvious convention gives equality predicates for all types: $E_{X\times Y}(\langle x, y\rangle, \langle x', y'\rangle)$ iff $E_X(x, x') \wedge E_Y(y, y')$, and so on. (Generally we will write $x = x'$ for $E_X(x, x')$.)

Note that every function and predicate symbol is typed: one type for the sorts of the arguments, and in addition a function symbol has a type for the sort of its "value". In the obvious way, this induces an assignment of types to all terms and formulae: we write $t\colon Y \to X$ and say t has *domain* Y and *codomain* X to mean that Y is the type giving the sorts of the free variables in t, and X is the sort of t. Similarly we write $\varphi\colon X$ (or $\varphi(x)$) and say φ *is over* X or *has type* X to mean that the free variables of φ have sorts given by the type X. (For example, "sentence" = "formula over 1"). *A technical point:* we want $\varphi\colon X \times Y$ (i.e. $\varphi = \varphi(x, y)$) to mean that the free variables of φ are *exactly* x, y of sorts X, Y, and not merely *among* x, y. Later we will want φ and ψ to have exactly the same free variables when we form, e.g., $\varphi \wedge \psi$. So that this is not too restrictive, we must be able to add "dummy" free variables to a formula. Perhaps the simplest way to do this is to add new function symbols to $\mathscr{L}$ corresponding to projections. (For example $\pi = \pi^{X\times Y}\colon X \times Y \to X$, $\pi(x, y) = x$; $!_Y = \pi_1^Y\colon Y \to 1$, $!_Y(y) = *$, where $*$ is a (the) free variable of type 1.) Then e.g. the formula $x_1 = x_2 \wedge x_2 = x_3$ would actually read $E_X(\pi_1(x_1, x_2, x_3), \pi_2(x_1, x_2, x_3)) \wedge \wedge E_X(\pi_2(x_1, x_2, x_3), \pi_3(x_1, x_2, x_3))$ where $\pi_1, \pi_2, \pi_3\colon X \times X \times X \to X$ are the evident projections.

The deduction rules and axioms are based on the standard natural deduction formulation of intuitionistic logic, as given in PRAWITZ [14], [15], with a few modifications. The basic rules are these:

$$(\wedge\mathrm{I})\ \frac{\varphi \quad \varphi'}{\varphi \wedge \varphi'} \qquad (\wedge\mathrm{E})_{\mathrm{L}}\ \frac{\varphi \wedge \varphi'}{\varphi} \qquad (\wedge\mathrm{E})_{\mathrm{R}}\ \frac{\varphi \wedge \varphi'}{\varphi'}$$

$$(\vee\mathrm{I})_{\mathrm{L}}\ \frac{\varphi}{\varphi \vee \varphi'} \qquad (\vee\mathrm{I})_{\mathrm{R}}\ \frac{\varphi'}{\varphi \vee \varphi'} \qquad (\vee\mathrm{E})\ \frac{\varphi \vee \varphi' \quad \begin{matrix}[\varphi]\\ \gamma\end{matrix} \quad \begin{matrix}[\varphi']\\ \gamma\end{matrix}}{\gamma}$$

$$(\supset\mathrm{I})\ \frac{\begin{matrix}[\varphi]\\ \varphi'\end{matrix}}{\varphi \supset \varphi'} \qquad (\supset\mathrm{E})\ \frac{\varphi \supset \varphi' \quad \varphi}{\varphi'}$$

$$(\forall\mathrm{I})\ \frac{\varphi(x)}{\forall\xi\varphi(\xi)} \qquad (\forall\mathrm{E})\ \frac{\forall\xi\varphi(\xi)}{\varphi(t)}$$

$$(\exists\mathrm{I})\ \frac{\varphi(t)}{\exists\xi\varphi(\xi)} \qquad (\exists\mathrm{E})\ \frac{\exists\xi\varphi(\xi) \quad \begin{matrix}[\varphi(x)]\\ \varphi'\end{matrix}}{\varphi'}$$

We add rules for $\top$ and $\bot$:

$$(!)\ \frac{\varphi}{\top_X} \qquad (\bot)\ \cancel{\frac{\varphi}{\bot_X}}\ \text{(reverse)}\ \frac{\bot_X}{\varphi}$$

where φ is an atomic formula over X different from $\top_X$ (in (!)) or from $\bot_X$ (in ($\bot$)), as appropriate. $\top_X$, $\bot_X$ are $\top$, $\bot$ with a dummy free variable of type X.

We add equality rules:

$$(=\mathrm{I})\ \frac{\top_X}{t = t} \quad \text{for any term } t\colon X \to Y,$$

$$(=\mathrm{E})\ \frac{s = s' \ldots t = t' \quad \varphi(s, \ldots, t)}{\varphi(s', \ldots, t')}$$

for any atomic formula φ over $X \times \ldots \times Y$, and any terms

$s, s'\colon X' \to X, \ldots, t, t'\colon Y' \to Y.$

(We will also denote the evident derived rules by (!), ($\bot$), (=E).)

In all of the rules except ($\forall$I), ($\exists$E), the premises and conclusion must be formulae over the same type. (And so, we may as well require that φ, ψ be over the same type if $\varphi \wedge \psi$, $\varphi \vee \psi$, $\varphi \supset \psi$ are to be wffs.)

In ($\forall$I), x must not occur in any assumption on which $\varphi(x)$ depends (this is standard) except possibly as a dummy free variable, in which case the dummy occurrences of x may be discharged.

In ($\exists$E), x must not occur in $\exists\xi\varphi$, in φ', or in any assumption other than φ on which the upper occurrence of φ' depends, except possibly as a dummy variable in the *upper* occurrence of φ', and the assumptions on which that occurrence depends, in which case the dummy occurrences of x may be discharged.

Note that we have in effect stratified derivations: a derivation $P\colon \Gamma \vdash \varphi$ must have all $\psi \in \Gamma$ and φ over the same type X: we say P is over X too. The rules ($\forall$I), ($\exists$E) provide the only way to change levels, by the discharge of dummy variables. (We will not usually explicitly show dummy variables, however; they can be filled in from the syntactic rules and the context.)

Finally, we denote by (id) the "rule" $\frac{\varphi}{\varphi}$ (rewriting φ) which should be understood as being merely the top occurrence of φ.