

(X)

$$\begin{array}{c}
 (3) \\
 \frac{rx = x' \wedge tx = y}{(\wedge E)} \\
 \\
 (2) \\
 \frac{(\wedge E) \frac{sy = t'x' \wedge \varphi(y)}{sy = t'x'}}{[t'x' = sy]} \quad \frac{(3) \frac{tx = y}{y = tx} (S) \quad \frac{(2) \frac{\varphi(y)}{\varphi(tx)}}{(\wedge I)}}{rx = x' \wedge \varphi(tx)} \quad \frac{(1) \frac{rx_1 = x' \wedge \varphi(tx_1)}{rx_1 = x'} (\wedge E)}{t'rx_1 = t'x'} (\text{ap}) \\
 \frac{P_I}{\exists \xi(r\xi = x' \wedge t\xi = y)} \quad \frac{(\exists I) \frac{rx = x' \wedge \varphi(tx)}{\exists \xi(r\xi = x' \wedge \varphi(t\xi))}}{\exists \xi(r\xi = x' \wedge \varphi(t\xi))} (\exists E) (3) \quad \frac{(1) \frac{t'rx_1 = t'x'}{\varphi(tx_1)}}{stx_1 = t'x'} (\text{id}') \quad \frac{(1) \frac{\varphi(tx_1)}{stx_1 = t'x' \wedge \varphi(tx_1)}}{(\wedge I)} (\wedge E) \\
 \frac{\exists \eta(s\eta = t'x' \wedge \varphi(\eta))}{\exists \xi(r\xi = x' \wedge \varphi(t\xi))} (\exists E) (2) \quad \frac{\exists \eta(s\eta = t'x' \wedge \varphi(\eta))}{\exists \xi(r\xi = x' \wedge \varphi(t\xi))} (\exists E) (1) \\
 \hline
 \exists \eta(s\eta = t'x' \wedge \varphi(\eta))
 \end{array}$$

It is easy to see this is equivalent to the identity (use (= Bed), (S Coh), (\wedge Exp) ($\exists E$ Simp), (\exists Exp)). This completes the proof. \square

As an immediate corollary to the theorem, we can characterize all the “left” structure of the base category \mathbf{T} of a hyperdoctrine \mathbf{P} over \mathbf{T} with the full Beck condition. As examples, we give the following. Suppose \mathbf{P} has the full Beck condition.

Corollary 1. $X \xrightarrow{r} Y$ is a monomorphism of \mathbf{T} iff there is a derivation in $\mathcal{T}_{\mathbf{P}}$:

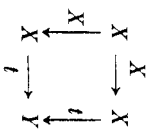
$$tr_0 = tr_1 \vdash x_0 = x_1.$$

Corollary 2. $E \xrightarrow{r} X \xrightarrow{s} Y$ is an equaliser diagram of \mathbf{T} iff there are derivations in $\mathcal{T}_{\mathbf{P}}$:

$$sr = tr \vdash \exists e(re = r), \quad re_0 = re_1 \vdash e_0 = e_1.$$

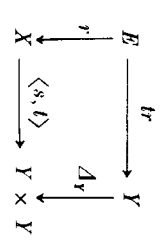
Corollary 3. If u_0, u_1 are free variables of type 1, then there is a derivation $\vdash u_0 = u_1$ (which is why we don't need many free variables of type 1).

Proofs. The first corollary is immediate from the theorem and the fact that $t: X \rightarrow Y$ is a monomorphism iff



is a pullback. (So the Beck condition for this diagram is equivalent to the existence of the above derivation (and of $x_0 = x_1, x_0 = x_1, x_0 = x_1 \vdash x_0 = x_1$!))

The second corollary is almost as immediate from the theorem and the fact that $E \xrightarrow{r} X \xrightarrow{s} Y$ is an equaliser iff



is a pullback. The Beck condition for this is equivalent to the existence of derivations $sr = y \wedge tx = y \vdash \exists e(re = x \wedge tre = y)$ and $re_0 = re_1, tre_0 = tre_1 \vdash e_0 = e_1$. Finally this is to the derivations of the statement of the corollary is an easy step.

And finally, since 1 is a terminal object,

