(X)

$$\dfrac{\dfrac{(3)}{rx = x' \wedge tx = y}\ (\wedge E)}{}$$

$$\dfrac{\dfrac{\dfrac{tx = y}{y = tx}\ (S) \qquad \dfrac{(2)}{\varphi(y)}\ (\wedge E)}{\varphi(tx)}\ (\text{sub}) \qquad \dfrac{(3)}{rx = x'}\ (\wedge E)}{rx = x' \wedge \varphi(tx)}\ (\wedge I)$$

$$\dfrac{rx = x' \wedge \varphi(tx)}{\exists\xi(r\xi = x' \wedge \varphi(t\xi))}\ (\exists I)$$

$$\dfrac{(2)}{\dfrac{sy = t'x' \wedge \varphi(y)}{\dfrac{sy = t'x'}{[t'x' = sy]}\ (S)}\ (\wedge E)}$$

$$P_I$$

$$\dfrac{\exists\xi(r\xi = x' \wedge t\xi = y) \qquad \exists\xi(r\xi = x' \wedge \varphi(t\xi))}{\exists\xi(r\xi = x' \wedge \varphi(t\xi))}\ (\exists E)\ (3)$$

$$\dfrac{\exists\eta(s\eta = t'x' \wedge \varphi(\eta)) \qquad \exists\xi(r\xi = x' \wedge \varphi(t\xi))}{\exists\xi(r\xi = x' \wedge \varphi(t\xi))}\ (\exists E)\ (2)$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{(1)}{rx_1 = x' \wedge \varphi(tx_1)}\ (\wedge E)}{rx_1 = x'}\ (\text{ap})}{t'rx_1 = t'x'}\ (\text{id})'}{stx_1 = t'x'} \qquad \dfrac{(1)}{\varphi(tx_1)}\ (\wedge E)}{stx_1 = t'x' \wedge \varphi(tx_1)}\ (\wedge I)$$

$$\dfrac{stx_1 = t'x' \wedge \varphi(tx_1)}{\exists\eta(s\eta = t'x' \wedge \varphi(\eta))}\ (\exists I)$$

$$\dfrac{\exists\xi(r\xi = x' \wedge \varphi(t\xi)) \qquad \exists\eta(s\eta = t'x' \wedge \varphi(\eta))}{\exists\eta(s\eta = t'x' \wedge \varphi(\eta))}\ (\exists E)\ (1)$$

It is easy to see this is equivalent to the identity (use (= Red), (S Coh), (∧ Exp
(∃ E Simp), (∃ Exp)). This completes the proof. □

As an immediate corollary to the theorem, we can characterise all the "left" stru
ture of the base category $\boldsymbol{T}$ of a hyperdoctrine $\boldsymbol{P}$ over $\boldsymbol{T}$ with the full Beck conditio
As examples, we give the following. Suppose $\boldsymbol{P}$ has the full Beck condition.

Corollary 1. *$X \xrightarrow{t} Y$ is a monomorphism of $\boldsymbol{T}$ iff there is a derivation in $T_{\boldsymbol{P}}$:*

$$tx_0 = tx_1 \vdash x_0 = x_1 .$$

Corollary 2. *$E \xrightarrow{r} X \underset{t}{\overset{s}{\rightrightarrows}} Y$ is an equaliser diagram of $\boldsymbol{T}$ iff there are derivatio
in $T_{\boldsymbol{P}}$:*

$$sx = tx \vdash \exists\varepsilon(r\varepsilon = x), \qquad re_0 = re_1 \vdash e_0 = e_1 .$$

Corollary 3. *If u_0, u_1 are free variables of type 1, then there is a derivation*

$$\vdash u_0 = u_1$$

(which is why we don't need many free variables of type 1).

Proofs. The first corollary is immediate from the theorem and the fact t
$t: X \to Y$ is a monomorphism iff

$$\begin{array}{ccc} X & \xrightarrow{X} & X \\ {\scriptstyle X}\downarrow & & \downarrow{\scriptstyle t} \\ X & \xrightarrow[t]{} & Y \end{array}$$

is a pullback. (So the Beck condition for this diagram is equivalent to the existe
of the above derivation (and of $x_0 = x_1$, $x_0 = x_1 \vdash x_0 = x_1$!))

The second corollary is almost as immediate from the theorem and the fact t
$E \xrightarrow{r} X \underset{t}{\overset{s}{\rightrightarrows}} Y$ is an equaliser iff

$$\begin{array}{ccc} E & \xrightarrow{tr} & Y \\ {\scriptstyle r}\downarrow & & \downarrow{\scriptstyle \Delta_Y} \\ X & \xrightarrow[\langle s, t\rangle]{} & Y \times Y \end{array}$$

is a pullback. The Beck condition for this is equivalent to the existence of derivati
$sx = y \wedge tx = y \vdash \exists\varepsilon(r\varepsilon = x \wedge tr\varepsilon = y)$ and $re_0 = re_1$, $tre_0 = tre_1 \vdash e_0 = e_1$. F
this to the derivations of the statement of the corollary is an easy step.

And finally, since 1 is a terminal object,

$$\begin{array}{ccc} 1 & \xrightarrow{\Delta_1} & 1 \times 1 \\ {\scriptstyle \Delta_1}\downarrow & & \downarrow{\scriptstyle 1 \times 1} \\ 1 \times 1 & \xrightarrow[1 \times 1]{} & 1 \times 1 \end{array}$$