Call this derivation P_0. Notice that $rx_1 = x' \wedge tx_1 = tx$ could be replaced with

$$\dfrac{\dfrac{\begin{matrix}(1)\\ rx = x' \wedge \varphi(tx)\end{matrix} \quad \begin{matrix}(3)\\ rx_1 = x' \wedge tx_1 = tx\end{matrix}}{rx = x' \wedge \varphi(tx) \wedge rx_1 = x' \wedge tx_1 = tx}\ (\wedge\ \mathrm{I})}{rx_1 = x' \wedge tx_1 = tx}\ (\wedge\ \mathrm{E})$$

since the latter reduces to the former by ($\wedge$ Red). We can expand this even further using the following:

Lemma 2. *Let Q_1 be the derivation*

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{rx = x' \wedge \varphi(tx) \wedge x_1 = x}{x_1 = x}\ (\wedge\ \mathrm{E}) \quad \dfrac{rx = x' \wedge \varphi(tx) \wedge x_1 = x}{rx = x'}\ (\wedge\ \mathrm{E})}{rx_1 = rx}\ (\mathrm{ap}) \quad \dfrac{rx = x' \wedge \varphi(tx) \wedge x_1 = x}{rx_1 = x'}\ (\wedge\ \mathrm{E})}{rx = x' \wedge \varphi(tx)}\ (\mathrm{T}) \quad \dfrac{rx = x' \wedge \varphi(tx) \wedge x_1 = x}{x_1 = x}\ (\wedge\ \mathrm{E})}{rx = x' \wedge \varphi(tx) \wedge x_1 = x}\ (\mathrm{ap}) \quad \dfrac{rx = x' \wedge \varphi(tx) \wedge x_1 = x}{tx_1 = tx}\ (\wedge\ \mathrm{E})}{rx = x' \wedge \varphi(tx) \wedge rx_1 = x' \wedge tx_1 = tx}\ (\wedge\ \mathrm{I})}{rx = x' \wedge \varphi(tx) \wedge rx_1 = x' \wedge tx_1 = tx}\ (\wedge\ \mathrm{I})$$

and Q_2 the derivation

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{rx = x' \wedge \varphi(tx) \wedge rx_1 = x' \wedge tx_1 = tx}{rx = x' \wedge \varphi(tx) \wedge rx_1 = x' \wedge tx_1 = tx}\ (\wedge\ \mathrm{E}) \quad rx = x'}{rx_1 = x'}\ (\mathrm{S}) \quad x' = rx}{rx_1 = rx}\ (\mathrm{T}) \quad \dfrac{rx = x' \wedge \varphi(tx) \wedge rx_1 = x' \wedge tx_1 = tx}{[rx_1 = rx \quad tx_1 = tx]}}{rx = x' \wedge \varphi(tx) \wedge rx_1 = x' \wedge tx_1 = tx}\ (\wedge\ \mathrm{E}) \quad P_E}{rx = x' \wedge \varphi(tx)} \quad x_1 = x}{rx = x' \wedge \varphi(tx) \wedge x_1 = x}\ (\wedge\ \mathrm{I})$$

Then the derivation

$$\begin{matrix} P \\ [rx = x' \wedge \varphi(tx) \wedge rx_1 = x' \wedge tx_1 = tx] \\ Q_2 \\ [rx = x' \wedge \varphi(tx) \wedge x_1 = x] \\ Q_1 \\ rx = x' \wedge \varphi(tx) \wedge rx_1 = x' \wedge tx_1 = tx \end{matrix}$$

is equivalent to

$$\begin{matrix} P \\ rx = x' \wedge \varphi(tx) \wedge rx_1 = x' \wedge tx_1 = tx \end{matrix}$$

for any P.

Proof. The proof of this fact is fairly straightforward, and so we will not write out the details here. Note, however, that the proof is simplified if one notices the following:

$$\begin{matrix} P & Q \\ [rx_0 = rx_1 & tx_0 = tx_1] \end{matrix}$$

$$\dfrac{\dfrac{P}{rx_0 = rx_1}\ P_E}{x_0 = x_1}\ (\mathrm{ap}) \quad \equiv \begin{matrix} P \\ rx_0 = rx_1 \end{matrix}$$

and

$$\begin{matrix} P & Q \\ [rx_0 = rx_1 & tx_0 = tx_1] \end{matrix}$$

$$\dfrac{\dfrac{Q}{tx_0 = tx_1}\ P_E}{x_0 = x_1}\ (\mathrm{ap}) \quad \equiv \begin{matrix} Q \\ tx_0 = tx_1 \end{matrix}$$

These equivalences are proved in the familiar way, using (= Exp) and (R Coh): replace x_1 with x_0, and note that $rx_0 = rx_0$ and $tx_0 = tx_0$ are isomorphic to $\top_X$ (so that the corresponding derivations must be equal). (Note also that there must be "dummy" assumptions on each RHS, as in ($\exists$ E Simp), to make sense of these equivalences.) □

Let P_1 be the derivation P_0 with the (discharged) assumption $rx_1 = x' \wedge tx_1 = tx$ replaced by the derivation

$$\dfrac{\dfrac{\begin{matrix}(1)\\ rx = x' \wedge \varphi(x)\end{matrix} \quad \begin{matrix}(3)\\ rx_1 = x' \wedge tx_1 = tx\end{matrix}}{rx = x' \wedge \varphi(tx) \wedge rx_1 = x' \wedge tx_1 = tx}\ (\wedge\ \mathrm{I})}{rx_1 = x' \wedge tx_1 = tx}\ (\wedge\ \mathrm{E})$$

$$\begin{matrix} Q_2 \\ rx = x' \wedge \varphi(tx) \wedge x_1 = x \\ Q_1 \\ rx = x' \wedge \varphi(x) \wedge rx_1 = x' \wedge tx_1 = tx \end{matrix}$$

Then P_1 is equivalent to P_0 (and so to the original composite derivation); we must show it equivalent to the identity.