

are pullbacks and that the Beck condition holds for each. Then

$$\begin{array}{ccc} X & \xrightarrow{t} & Y \\ r \downarrow & & \downarrow s \\ X' & \xrightarrow{t'} & Y' \end{array}$$

is a pullback and the Beck condition holds for any φ over Y of the form $t^*\psi$ for ψ over Z (viz. $\Sigma_! t^* \varphi \cong t^* \Sigma_! \varphi$ if $\varphi = t_1^* \psi$).

Proof trivial.

Corollary 1. For (D) as in the theorem, the Beck condition for $\top_X$ holds for

$$\begin{array}{ccc} X & \xrightarrow{\quad} & X \\ X \downarrow & & \downarrow \langle r, t \rangle \\ X & \xrightarrow{\quad} & X' \times Y \\ & & \downarrow \langle r, t \rangle \end{array}$$

(viz. $\top_X \cong \langle r, t \rangle^* \Sigma_{\langle r, t \rangle}^* \top_X$).

Proof. Consider

$$\begin{array}{ccccc} X & \xrightarrow{\quad} & X & \xrightarrow{t'r} & Y' \\ X \downarrow & & \downarrow \langle r, t \rangle & & \downarrow A_{Y'} \\ X & \xrightarrow{\quad} & X' \times Y & \xrightarrow{t' \times s} & Y' \times Y' \\ & & \downarrow \langle r, t \rangle & & \end{array}$$

note that $(t' \times s) \langle r, t \rangle = \langle t'r, t'r \rangle$, and that $\top_X = (t'r)^* \top_{Y'}$.

Lemma 4. For any $t: X \rightarrow Y$, if

$$(D_0) \quad \begin{array}{ccc} X & \xrightarrow{\quad} & X \\ X \downarrow & & \downarrow t \\ X & \xrightarrow{\quad} & Y \end{array}$$

is a pullback, then so is

$$(D_1) \quad \begin{array}{ccc} X & \xrightarrow{t} & Y \\ A_X \downarrow & & \downarrow A_Y \\ X \times X & \xrightarrow{t \times t} & Y \times Y \end{array}$$

Moreover, if the Beck condition (for φ over X) holds for (D_0) , then it does also (respectively for any ψ over Y so that $t^* \psi = \varphi$) for (D_1) .

Proof. Immediate from Lemma 2.

Corollary 2. For (D) as in the theorem, $(\langle r, t \rangle \times \langle r, t \rangle)^* E_{X' \times Y} \cong E_X$.

Proof. $E_Z =_{df} \Sigma_{A_Z} \top_Z$.

8.2. Suppose T_P satisfies the condition of the theorem. We will show that from this it follows that $\Sigma_! t^* \varphi \cong t^* \Sigma_! \varphi$ for any $\varphi \in |P(Y)|$; the dual result will follow immediately.

Before we begin, however, we must mention an equivalence scheme we shall need

Lemma 1. If no assumption in P_1 is discharged by the use of $(\exists E)$ in question, the following derivations are equivalent:

$$(\exists E \text{ Simp}) \quad \frac{P \quad P_1 \quad \frac{\exists \xi \varphi \quad \theta \quad P_1}{\theta}}{\theta} \equiv \theta$$

Remark. We could have included this in Corollary 2 of § 2, but as we have not yet needed it, and as the proof is simplified by the Theorems of § 6, we have left it instead until now.

Proof. First, we should clarify what is meant: P_1 is a derivation of θ from assumptions A_1 , say, which do not include φ and do not have x as a nondummy free variable. P is a derivation of $\exists \xi \varphi$ from assumptions A , say. So the LHS is a derivation of θ from $A \wedge A_1$ (really the set $A \cup A_1$), and the RHS is supposed then to be the derivation P_1 with "dummy" assumption A — we can think of it as $A \wedge A_1 \xrightarrow{\text{proj}} A_1 \xrightarrow{P_1} \theta$. Suppose $\varphi \in |P(X \times Y)|$, $A \xrightarrow{P_1} \theta$ is in $P(Y)$, and $X \times Y \xrightarrow{\pi} Y$. To prove the lemma it suffices to show that the canonical morphism $\Sigma_! \varphi \wedge A \xrightarrow{P_1} \theta$, induced by $\varphi \wedge \pi^* A \xrightarrow{\text{proj}} \pi^* A \xrightarrow{\pi^* P_1} \pi^* \theta$ under $\Sigma_! \dashv \pi^*$, is in fact $\Sigma_! \varphi \wedge A \xrightarrow{\text{proj}} A \xrightarrow{P_1} \theta$. Now $\bar{P}$ is defined using the cartesian closedness of the fibres (and Frobenius Reciprocity):

$$\frac{\varphi \wedge \pi^* A \xrightarrow{\text{proj}} \pi^* A \xrightarrow{\pi^* P_1} \pi^* \theta}{\varphi \rightarrow (\pi^* A \rightarrow \pi^* \theta) \xrightarrow{\sim} \pi^*(A \supset \theta)} \quad \frac{\Sigma_! \varphi \rightarrow (A \supset \theta)}{\Sigma_! \varphi \wedge A \rightarrow \theta} \quad \bar{P}_1$$

So the result is an immediate consequence of the fact that in any cartesian close category with a terminal object 1, the bijection

$$A \wedge B \xrightarrow{f} C$$

$$\frac{A \wedge B \xrightarrow{f} C}{A \xrightarrow{r_1} (B \supset C)}$$

restricts to a bijection

$$\frac{A \wedge B \xrightarrow{\text{proj}} B \xrightarrow{f} C}{A \xrightarrow{r_1} 1 \xrightarrow{r_1} (B \supset C)} \quad \square$$

We may also remark that there is an analogous scheme for $\vee$:

$$(\vee E \text{ Simp}) \quad \frac{P \quad P_1 \quad P_1 \quad \frac{\varphi \vee \psi \quad \theta \quad P_1}{\theta} \equiv \theta}{\theta}$$