

Definition. A *locally closed category* is a category $\mathcal{B}$ with finite limits and finite coproducts (including a 0), such that for any object X of $\mathcal{B}$, $\mathcal{B}/X$ is cartesian closed.

I believe the following observation is due independently to LAWVERE and BENABOU — a proof appears in FREYD [2] (Proposition 1.34).

Proposition. If $\mathcal{B}$ is locally closed, then $\mathcal{B}$, as a $\mathcal{B}$ -category, is a *hyperdoctrine*. Conversely, if $\mathcal{B}$ has finite limits, and $\mathcal{B}$, as a $\mathcal{B}$ -category, is a *hyperdoctrine*, then $\mathcal{B}$ is locally closed. In fact, if $\mathcal{B}$ has finite limits and finite coproducts, and $\mathcal{B}$, as a $\mathcal{B}$ -category, is locally small, then $\mathcal{B}$ is locally closed.

Remark. $\mathcal{B}$, as a $\mathcal{B}$ -category, is the indexed category with fibres $\mathcal{B}(X) = \mathcal{B}/X$, the comma category. This is locally small iff for each X , $\mathcal{B}/X$ is cartesian closed, iff for each t , t^* has a right adjoint Π_t . (See, e.g., PARÉ-SCHUMACHER [13].) (The presence of coproducts is irrelevant to all this — we include them only because our logic includes $\vee$.)

Remark. If $\mathcal{B}$ is both a *logos* and *locally closed*, (e.g. a *topos*) then there is an *indexed functor* between these hyperdoctrines, $\text{Im}: \mathcal{B} \rightarrow \text{Sub}_{\mathcal{B}}$, taking a morphism to its image. This preserves $\wedge, \vee, \top, \perp, ()^*, \text{ and } \Sigma$; it also preserves Π (equivalently $\rhd$) and so is a *hyperdoctrine morphism* iff for any $t: X \rightarrow Y$, Π_t preserves regular epis. In a *topos*, this is a weak version of the axiom of choice. (See JOHNSTONE [6] § 5.2; also SEELY [18] for details.)

7.3. Our third example just misses being a hyperdoctrine: The base category is $\mathcal{G}/d$, the category of all groupoids. (A groupoid is a category all of whose morphisms are isomorphisms.) Given a groupoid X , the fibre $\mathcal{P}(X) = \mathcal{G}/\mathcal{E}X$ is the category of functors $X \rightarrow \mathcal{G}/\mathcal{E}$. (If X is a group — i.e., has only one object — then $\mathcal{G}/\mathcal{E}X$ is perhaps better known as the category of permutation representations of X .) $\mathcal{G}/\mathcal{E}X$ is a topos, and so is certainly locally closed. Given a functor (groupoid homomorphism) $X \xrightarrow{t} Y$, $t^*: \mathcal{G}/\mathcal{E}Y \rightarrow \mathcal{G}/\mathcal{E}X$ is defined by composition with t . Σ_t, Π_t are defined as left and right Kan extensions (c.f. MACLANE [11]). (For groups Σ_t, Π_t is the “induced representation” of Y ; we may call Π_t the “dual induced representation”.)

For the benefit of the non-category theorist (and because we will need this later) I will explicitly calculate Σ_t, Π_t for $\varphi: X \rightarrow \mathcal{G}/\mathcal{E}$. To simplify notation, let us suppose X and Y are groups; denote their unique objects by X and Y respectively, and write “ $z \in Z$ ” for “ z is a morphism of Z ”. In this case $\varphi: X \rightarrow \mathcal{G}/\mathcal{E}$ can be represented as $\langle \varphi, q_x \rangle_{x \in X}$ where φ is a set (it is $q(X)$), and for each $x \in X$, q_x is a permutation of φ (satisfying the usual equation $q_{x_0 x_1^{-1}} = q_{x_0} q_{x_1}^{-1}$, so that the q_x define an X action on φ).

Then $\Sigma_t \varphi$ is, as a set, the “orbit set” of $Y \times \varphi$, i.e. $Y \times \varphi / \sim$ for $(y_0, a_0) \sim (y_1, a_1)$ iff for some $x \in X$ $(y_1, a_1) = (y_0 x, a_1 x^{-1} a_0) = a_1 x (y_0, a_0)$. (This defines the action of X on $\Sigma_t \varphi$ also.) For groupoids the idea is similar. (Details in SEELY [18].)

Proposition. (i) The groupoid representation structure satisfies *Frobenius Reciprocity*. (ii) It does satisfy the Beck condition for pullbacks of types (a) and (b) (cf. § 3), but not, in general, for pullbacks of type (c). (iii) The Beck condition is preserved under products, in the case where φ is E_Y (the case used in § 8).

Remarks. (i) is proven in LAWVERE [10], (where LAWVERE also seems to suggest that (ii) is false, in that he suggests that his “meagre theorems apparently do not

hold in [this structure]”, where these theorems depend only on the Beck condition for pullbacks of types (a) and (b)). (ii) and (iii) may in fact be proven by a direct “diagram chase”; details are given in SEELY [18].

Among other things, this example shows that the operation (R Coh) is independent of the usual expansions, reductions, and permutations. Its logic is perfectly well behaved as far as the equality-free part is concerned, but the logic of equality does show some curiosities. Clearly $t = t$ is quite different from $\top$, and so can be an important assumption in a derivation, for instance. Also $x = y$ and $\langle x, x \rangle = \langle y, y \rangle$ behave quite differently. However, some things remain: the rules (sub) and (R) (though (R) is not an isomorphism), and the equivalences ($=$ Red), ($=$ Exp) all hold in $\mathcal{G}/d$.

§ 8. The Beck Condition

Most of the structure in the definition of a hyperdoctrine is (from the logical point of view) more or less self-evident—that is, it is clear why the definition must impose such conditions. The only question one might pose is “why do we need the Beck condition?” Part of the answer is already given to us: to even begin interpreting the categorical structure logically, the Beck condition is needed for pullbacks of the three types given in § 3. Type (a) allows us to interpret Σ_t, φ as $\exists \xi (t\xi = x \wedge \varphi(\xi))$ —otherwise it would just be some formula about which we knew little. Type (b) allows us to treat types as if they were sorts, especially with respect to equality. It also gives a result analogous to Frobenius Reciprocity, viz. that substitution is well-defined: for

$$\begin{array}{ccc} A \times X & \xrightarrow{A \times t} & A \times Y \\ \downarrow \pi_X & & \downarrow \pi_Y \\ X & \xrightarrow{t} & Y \end{array}$$

becomes the isomorphism $t^* \Sigma_t \varphi \cong \Sigma_{\pi_X} (A \times t)^* \varphi$, where

$$[\exists x \varphi(x, y)](t) = \exists x \varphi(x, tx)$$

and φ is over $A \times Y$. Type (c) gives the isomorphism $(x = x) \cong \top_X$ (and in practice allows us to treat Δ_X as the diagonal morphism it is). There is one further condition we need: the preservation of the Beck condition under products. This amounts to the following: if for all φ over Y , $\exists \xi (r\xi = x \wedge \varphi(t\xi)) \cong \exists \eta (s\eta = t'x \wedge \varphi(\eta))$ then for all η over $Y \times Z$,

$$\exists \xi \exists \zeta (r\xi = x \wedge \zeta = z \wedge \psi(t\xi, \zeta)) \cong \exists \eta \exists \zeta (s\eta = t'x \wedge \zeta = z \wedge \psi(\eta, \zeta)).$$

(Which says that we may suppress free variables in considering the Beck condition. This would seem justified without further comment.)

There is one remark we should perhaps make here: if we wished, we could do without the Beck condition for pullbacks of type (c). Of course, we would then have to drop (R Coh) from the operations of § 2, and pay very close attention to the assumptions of our derivations (not overlooking any of the form $t = t$) and to any use of the diagonal (not confusing the formulae $\langle t, t \rangle = \langle t', t' \rangle$ and $t = t'$). An example of structure of this sort was seen in § 7. However, we shall continue to regard (R Coh) as a desirable equivalence, and so shall continue to suppose Beck for type (c).