

$\exists \xi(\xi = x_0 \wedge \xi = x_1 \wedge \top x)$ (i.e., $\Sigma_x, \top x$), so “new” and “old” equality agree. Finally, in any hyperdoctrine the Beck condition for type (b) pullbacks implies that the “equality” meta-notion we defined was correct: equality over the sort $X \times Y$ of T_P and “meta-equality” over the type $X \times Y$ of T agree. (See SEELY [18] for details.)

So we have proved the theorem, and even more, once we note that the operation (functor) $\varphi \mapsto \bar{\varphi}$, $P \mapsto \bar{P}$ is full and faithful (up to equivalence of derivations). (This justifies the assertion that T and T_P are equivalent as categories.)

Theorem. *If P is a hyperdoctrine, then P_{TP} is an equivalent hyperdoctrine.*

Proof. First, note that there are canonical functors $P \xrightarrow{j} P_{TP}$, $T \xrightarrow{i} T_{TP}$, which are fully faithful. These functors are also essentially surjective – the proof of this fact also shows that P and P_{TP} are equivalent as hyperdoctrines, since $()^*$ and its adjoints Σ, Π are preserved up to isomorphism. That i is essentially surjective is seen easily: the only “new” objects of T_{TP} correspond to the types of T_P which are not sorts, and these are isomorphic to obvious sorts (or objects of T_{TP}); morphisms of T_{TP} (“terms”) are treated similarly. The “propositional” part of the proof that j is essentially surjective is analogous. Since we added rules to T_P to make $t^*\varphi$ and $\varphi(t)$ isomorphic, and treated the quantifiers similarly, j must be essentially surjective. But even more can be asserted: the adjoints agree up to isomorphism. By a theorem of LAWVERE in [10] $\Sigma_i \varphi \cong \Sigma_{\pi_i}(E_i \wedge \pi_i^* \varphi)$ for any hyperdoctrine with the Beck condition for pullbacks of type (a), where $\varphi \in |\mathcal{P}(X)|$ and $\pi_i: X \times Y \rightarrow X, \pi_i^*: X \times Y \rightarrow X$, $t: X \rightarrow Y$ in T , and $E_i =_{\alpha} (t \times Y)^* E_Y$. One can also show (under the same assumptions) $\Pi_i \varphi \cong \Pi_{\pi_i}(E_i \rhd \pi_i^* \varphi)$. Using this, it is easy to see that the adjoints to t^* agree – the left hand side (of these isomorphisms) giving the adjoint in P , and the right hand side giving that in P_{TP} . This completes the proof.

Remark. We have not yet defined the notion of a “morphism of hyperdoctrines”, but it is clear that what we would want is the following. Suppose P_0 (over T_0), P_1 (over T_1) are hyperdoctrines. Then a morphism F from P_0 to P_1 is given by a functor $T_0 \xrightarrow{F_0} T_1$, and for each $X \in |T_0|$, a functor $F_1(X): P_0(X) \rightarrow P_1(F_0 X)$, with the evident preservation properties: F_0 preserves finite products; for each X , $F_1(X)$ preserves the locally closed structure of $P_0(X)$; and for any $X \xrightarrow{t} Y$ in T_0 , the following commute (up to isomorphism):

$$(i) \quad \begin{array}{ccc} P_0(X) & \xrightarrow{\Sigma_i} & P_0(Y) \\ F_1(X) \downarrow & & \downarrow F_1(Y) \\ P_1(F_0 X) & \xrightarrow{\Sigma_{F_0 i}} & P_1(F_0 Y) \end{array}$$

$$(ii) \quad \begin{array}{ccc} P_0(X) & \xleftarrow{t^*} & P_0(Y) \\ F_1(X) \downarrow & & \downarrow F_1(Y) \\ P_1(F_0 X) & \xleftarrow{(F_0 t)^*} & P_1(F_0 Y) \end{array}$$

$$(iii) \quad \begin{array}{ccc} P_0(X) & \xrightarrow{\Pi_i} & P_0(Y) \\ F_1(X) \downarrow & & \downarrow F_1(Y) \\ P_1(F_0 X) & \xrightarrow{\Pi_{F_0 i}} & P_1(F_0 Y) \end{array}$$

This just corresponds to the notion of an interpretation of T_{P_0} in T_{P_1} , SHOENFELD [20] (modified to suit our treatment of first order logic, of course.) (If we think of P as a (pseudo) functor $T^{op} \rightarrow \mathcal{Cat}$, then (ii) is just the condition that $F_1: P_0 \rightarrow P_1 F_0$ is natural.)

If we then define the category $\mathcal{Hyp}$ of hyperdoctrines and morphisms of hyperdoctrines, and the category $\mathcal{Hdg}$ of theories and interpretations of theories, then we have shown

Theorem. *The categories $\mathcal{Hyp}$ and $\mathcal{Hdg}$ are equivalent.*

(In § 8 we will refine the equivalence to one dealing with hyperdoctrines satisfying the Beck condition locally.)

§ 7. Examples

We shall briefly describe two well known examples of hyperdoctrines, and a structure not quite a hyperdoctrine.

7.1 Definition. A *logos* is a category L with finite limits and stable image factorisations, such that every object has a minimal subject, every pair of subobjects has a union, and for every morphism $X \xrightarrow{t} Y$ and subobject $X' \rightarrow X$ there is a subobject $\Pi_t X' \rightarrow Y$ maximal among subobjects of Y whose pullbacks along t lie in X' .

There are many examples of logoi: for instance, any topos is a logos. (This is no more than the truism that first order logic is part of higher order logic, in view of the following.)

Any logos L induces an indexed category Sub_L (over L) in the obvious way by taking the poset of subobjects of X as the fibre over X , and defining t^* as pullback along t . The following is well known (see REYES [16]):

Proposition. *If L is a logos, then Sub_L is a hyperdoctrine. Conversely, if L has finite limits and Sub_L is a hyperdoctrine, then L is a logos.*

So we can identify logoi as certain hyperdoctrines, whose fibres are posets. In fact, if $\mathcal{Log}$ is the category of logoi (and structure preserving functors) and $\mathcal{Hyp}$ the full subcategory of $\mathcal{Hyp}$ of hyperdoctrines with poset fibres (“po-hyperdoctrines”), then:

Proposition. *$\mathcal{Log}$ is a reflective subcategory of $\mathcal{Hyp}$.*

(The reflection is given by the construction of the “Lindenbaum-Tarski-category” of the theory corresponding to a po-hyperdoctrine. See SEELY [18] for details.)

7.2. Our first example of a hyperdoctrine dealt with the notion of “propositional function”; the next will concern that of “indexed family”.