$$\begin{tikzcd}
X' \times Y' \times X' \times Y' \arrow[r, "s \times t \times s' \times t'"] \arrow[d, "\pi'_{12}"] \arrow[d, shift left=6, "\pi'_{13}"] \arrow[d, shift right=6, "\pi'_{24}"] \arrow[d, shift right=12, "\pi'_{34}"] & X \times Y \times X \times Y \arrow[d, "\pi_{12}"] \arrow[d, shift left=6, "\pi_{13}"] \arrow[d, shift right=6, "\pi_{24}"] \arrow[d, shift right=12, "\pi_{34}"] \\
X' \times Y' \quad X' \times X' \quad Y' \times Y' \quad X' \times Y' & X \times Y \quad X \times X \quad Y \times Y \quad X \times Y
\end{tikzcd}$$

(Note the four evident commutative squares.)

We have shown that $T_{\boldsymbol{P}}$ allows all the derivations we would expect of a theory in LPCE; we can do more, though, for it also has all the equivalences we would expect from the operations of § 2. As examples, we shall prove the only two that might pose any problems: (= Red) and (= Exp).

(= Red) To see this equivalence holds in $T_{\boldsymbol{P}}$, it is sufficient to show (for $\varphi \in |\boldsymbol{P}(X)|$)

$$\varphi \xrightarrow[\langle 1_\varphi, !_\varphi \rangle]{} \varphi \wedge \top_X \xrightarrow[1_\varphi \wedge \eta_{\top_X}]{} \varphi \wedge \Lambda^*_X E_X \xrightarrow[\Lambda^*_X s]{} \varphi$$

is the identity, where s is the morphism defined above (essentially (sub)), and $E_X = \Sigma_{\Delta_X} \top_X$. Using the definition of s and the triangle equality $\Delta^*_X \varepsilon \cdot \eta \Delta^*_X = \mathrm{id}$, this reduces to showing the square (*) commutes:

$$\begin{tikzcd}
 & \varphi \wedge \top_X \arrow[r, "1_\varphi \wedge \eta_{\top_X}"] \arrow[d, "(*)_3"] & \varphi \wedge \Lambda^*_X \Sigma_{\Delta_X} \top_X \arrow[d, "\eta_{\Delta^*_X}(\pi^* \varphi \wedge E_X)"] \arrow[dd, bend left=60, "\mathrm{id}"] \\
\varphi \arrow[ur, bend left=60, "\langle 1_\varphi, !_\varphi \rangle"] \arrow[r, "\eta_\varphi"] \arrow[d, "\langle 1_\varphi, !_\varphi \rangle"] \arrow[d, shift right=6, "(*)"] & \Lambda^*_X \Sigma_{\Delta_X} \varphi \arrow[r, "\Lambda^*_X \Sigma_{\Delta_X}(1_\varphi, \eta_{\top_X})"] \arrow[d, "\Lambda^*_X f"] & \Lambda^*_X \Sigma_{\Delta_X} \Delta^*_X (\pi^* \varphi \wedge E_X) \arrow[d, "(*)_4"] \arrow[dl, "(*)_2 \ \Lambda^*_X \varepsilon_{\pi^* \varphi \wedge E_X}"] \\
\varphi \wedge \top_X \arrow[r, "1_\varphi \wedge \eta_{\top_X}"] & \varphi \wedge \Lambda^*_X \Sigma_{\Delta_X} \top_X \arrow[ur, bend right=60] &
\end{tikzcd}$$

f is the canonical map, so that $(*)_2$ commutes by definition; $(*)_3$ commutes by the naturality of η, and $(*)_4$ by the triangle equality. □

(= Exp) We must show

$$E_X \xrightarrow[\langle !_{E_X}, !_{E_X}, 1_{E_X} \rangle]{} \top_{X^2} \wedge \top_{X^2} \wedge E_X \xrightarrow[\pi_1^* \eta_{\top_X} \wedge \pi_1^* \eta_{\top_X} \wedge 1_{E_X}]{}$$
$$\to \pi_1^* \Lambda^*_X E_X \wedge \pi_1^* \Lambda^*_X E_X \wedge E_X \xrightarrow[\nabla^* s]{} E_X$$

is the identity, where $\nabla\colon X \times X \to X \times X \times X \times X$ is $\langle 1_X, 1_X, 1_X \rangle \times 1_X$ (i.e. "$\langle x_0, x_1 \rangle \mapsto \langle x_0, x_0, x_0, x_1 \rangle$".) In fact, we will show the corresponding morphism $\top_X \to \Lambda^*_X E_X$ is the unit $\eta_{\top_X}$. By a process similar to that used for (= Red), we can reduce this to showing

$$\begin{tikzcd}
\top_X \arrow[r, "\langle \eta^1_{\top_X}, \eta^1_{\top_X} \rangle"] \arrow[d, "\Lambda^*_X \eta^2_{\top_X^2}"] & \Lambda^*_X E_X \wedge \Lambda^*_X E_X \arrow[d, "\Lambda^*_X \Lambda^*_{X^2} \varepsilon_{\pi_{13}^* E_X \wedge \pi_{24}^* E_X}"] \\
\Lambda^*_X \Lambda^*_{X^2} E_{X^2} \arrow[r, "\Lambda^*_X \Lambda^*_{X^2} \Sigma_{\Delta_{X^2}} (\pi_1^* \eta^1_{\top_X}, \pi_2^* \eta^1_{\top_X})"'] & \Lambda^*_X \Lambda^*_{X^2} \Sigma_{\Delta_{X^2}} \Delta^*_{X^2} (\pi_{13}^* E_X \wedge \pi_{24}^* E_X)
\end{tikzcd}$$

commutes, where η^i is the unit of $\Sigma_{\Delta_{X^i}} \vdash \Lambda^*_{X^i}$ $(i = 1, 2)$ ($X^2 = X \times X$, of course! (The rest of the notation is standard.) By considering the various component $\top_X \to \Lambda^*_X E_X$, and recalling that "the counit of a product is the product of the counits" this reduces in turn to considering

$$\begin{tikzcd}
\top_X \arrow[r, dashed, "\eta^1_{\top_X}"] \arrow[d, "\Lambda^*_X \eta^2_{\top_X}"] & \Lambda^*_X E_X \\
\Lambda^*_X \Lambda^*_{X^2} E_{X^2} \arrow[r, dashed, "\Lambda^*_X \Lambda^*_{X^2} \Sigma_{\Delta_{X^2}} \pi_1^* \eta^1_{\top_X}"'] & \Lambda^*_X \Lambda^*_{X^2} \Sigma_{\Delta_{X^2}} \Lambda^*_{X^2} \pi_1^* E_X \arrow[u, "\Lambda^*_X \Lambda^*_{X^2} \varepsilon^2_{\pi_{13}^* E_X}"']
\end{tikzcd}$$

However, this commutes (using the triangle equality $\Lambda^*_{X^2} \varepsilon^2 \cdot \eta^2 \Lambda^*_{X^2} = \mathrm{id}$) by t naturality of η^2:

$$\begin{tikzcd}
\top_{X^2} \arrow[r, "\pi_1^* \eta^1_{\top_X}"] \arrow[d, "\eta^2_{\top_{X^2}}"] & \pi_1^* \Lambda^*_X E_X \arrow[d, "\eta^2_{\pi_1^* \Delta_X^* E_X}"] \\
\Lambda^*_{X^2} \Sigma_{\Delta_{X^2}} \top_{X^2} \arrow[r, "\Lambda^*_{X^2} \Sigma_{\Delta_{X^2}} \pi_1^* \eta^1_{\top_X}"'] & \Lambda^*_{X^2} \Sigma_{\Delta_{X^2}} \pi_1^* \Lambda^*_X E_X
\end{tikzcd}$$

So we have the following:

Theorem. *If $\boldsymbol{P}$ is a hyperdoctrine, then $T_{\boldsymbol{P}}$, as constructed above, is a theory in* LP *in that every deduction rule of* LPCE *is derivable in $T_{\boldsymbol{P}}$, and all the equivalences genere by the operations of § 2, are satisfied (and in fact, are identities).*

§ 6. Equivalences

What happens if we iterate these constructions? Nothing new:

Theorem. *If T is a theory in* LPCE, *then $T_{\boldsymbol{P}_T}$ is equivalent to T in the following s for every formula φ of T, there is an (atomic) formula $\bar{\varphi}$ of $T_{\boldsymbol{P}_T}$ canonically induced b Also, for every derivation $P\colon \varphi \vdash \psi$ of T, there is a derivation (rule of inference, in $\bar{P}\colon \bar{\varphi} \vdash \bar{\psi}$ of $T_{\boldsymbol{P}_T}$ canonically induced by it. Moreover, every formula φ' of $T_{\boldsymbol{P}_T}$ is ically equivalent to $\bar{\varphi}$ for some φ of T (in fact, isomorphic, in the sense of the catego structure on $T_{\boldsymbol{P}_T}$), and for every derivation $P'\colon \varphi' \vdash \psi'$ of $T_{\boldsymbol{P}_T}$ there is a derive $\bar{P}\colon \bar{\varphi} \vdash \psi$ induced by some $P\colon \varphi \vdash \psi$ in T, where $\bar{\varphi}$ (respectively $\bar{\psi}$) is logically e alent (in fact isomorphic) to φ' (respectively ψ'). (P' and $\bar{P}$ are also "isomorphic".)*

(What we shall actually prove is: T and $T_{\boldsymbol{P}_T}$ are equivalent as categorical struct This implies they are logically equivalent theories.)

Proof. Given φ in T, by construction $\varphi \in |\boldsymbol{P}_T|$ and so is a predicate symbol of $\bar{\varphi}$ is just this predicate symbol with the suitable free variables to make it a well-fo (atomic) formula. Given $P\colon \varphi \vdash \psi$ in T, P is (or rather, its equivalence class morphism of $\boldsymbol{P}$, and so is a rule of inference of $T_{\boldsymbol{P}_T}$; $\bar{P}$ is the corresponding (tr derivation.

In defining $T_{\boldsymbol{P}}$ for any hyperdoctrine $\boldsymbol{P}$, we added enough rules to enable conclude that every formula is equivalent to an atomic formula. Moreover, by con ing the adjointness property of equality, we see that $x_0 = x_1$ is isomorpl