This is not quite enough, however—intuitively, we expect $\varphi \times \psi$ to be the same as $\varphi \wedge \psi$, but $\varphi \times \psi$ is an atomic formula, whereas $\varphi \wedge \psi$ certainly is not. It is easy to see that there is a derivation $\varphi \times \psi \vdash \varphi \wedge \psi$ (viz.

$$\dfrac{\begin{array}{cc}\varphi \times \psi & \varphi \times \psi \\ P_{\pi_1} & P_{\pi_2} \\ \varphi & \psi\end{array}}{\varphi \wedge \psi}$$

where P_{π_1}, P_{π_2} are the derivations (rules of inference, in fact) corresponding to the projections $\pi_1 \colon \varphi \times \psi \to \varphi$, $\pi_2 \colon \varphi \times \psi \to \psi$). But there is no reason to suppose that there is an inverse derivation $\varphi \wedge \psi \vdash \varphi \times \psi$, so if we want these to be equivalent, we must bodily add such a rule to T. Moreover, if we want this equivalence to be of the nature of an isomorphism, we must add reductions to the effect that the two "compositions" $\varphi \wedge \psi \to \varphi \times \psi \to \varphi \wedge \psi$ and $\varphi \times \psi \to \varphi \wedge \psi \to \varphi \times \psi$ both reduce to the identity. So we add to T the rule $\dfrac{\varphi \wedge \psi}{\varphi \times \psi}$ ($\times$ I), the reduction

($\times$ Red)
$$\dfrac{\dfrac{\dfrac{P}{\varphi \wedge \psi}}{\varphi \times \psi}\quad \dfrac{\dfrac{P}{\varphi \wedge \psi}}{\varphi \times \psi}}{\begin{array}{cc}P_{\pi_1} & P_{\pi_2} \\ \varphi & \psi\end{array}}}{\varphi \wedge \psi} \quad \Rightarrow \dfrac{P}{\varphi \wedge \psi}$$

and the expansion

($\times$ Exp)
$$\dfrac{P}{\varphi \times \psi} \Rightarrow \dfrac{\dfrac{\dfrac{P}{\varphi \times \psi}}{\varphi \times \psi}\quad \dfrac{\dfrac{P}{\varphi \times \psi}}{\varphi \times \psi}}{\begin{array}{cc}P_{\pi_1} & P_{\pi_2} \\ \varphi & \psi\end{array}}}{\varphi \wedge \psi}$$

We treat the remaining logical connectives and quantifiers in a similar fashion, so that $\times = \wedge$, $\amalg = \vee$, $\exp = \supset$, $\Pi_\pi = \forall$, $\Sigma_\pi = \exists$, $0 = \bot$, $1 = \top$. (By $\Pi_\pi = \forall$ we mean $\forall$ is given by the "Π-image" along a projection; similarly for $\Sigma_\pi = \exists$.) Finally, we want t^* to mean substitution of t, so we must add the rules

$$\dfrac{\varphi(t)}{t^*\varphi}\ (^* \text{ I}), \qquad \dfrac{t^*\varphi}{\varphi(t)}\ (^* \text{ E}),$$

the reduction

(* Red)
$$\dfrac{\dfrac{\dfrac{P}{\varphi(t)}}{t^*\varphi}}{\varphi(t)} \Rightarrow \dfrac{P}{\varphi(t)},$$

and the expansion

(* Exp)
$$\dfrac{P}{t^*\varphi} \Rightarrow \dfrac{\dfrac{P}{t^*\varphi}}{\varphi(t)}$$

The equality predicate of sort X, for any $X \in |T|$, is $\Sigma_{\Delta_X} \top_X$. (By an abuse of not tion that is justified by the above, the terminal object of $\boldsymbol{P}(X)$ will be denoted $\top$ Similarly, products, coproducts, exponentiation, and the initial object in $\boldsymbol{P}(X)$ w henceforth be denoted $\wedge$, $\vee$, $\supset$, and $\bot_X$ respectively. The categorical structure of on the other hand, will retain the more customary notation.)

We must now make sure that all this is justified: that in fact $\varphi \wedge \psi$ has the t logical properties we expect of $\varphi \wedge \psi$, and so on for the rest of the structure. This well known for the propositional connectives: see SEELY [19]. That the (ordina quantifiers $\exists, \forall$ are given by the adjoints to projections—if φ is over $X \times Y$, th $\forall \xi \varphi$ (respectively $\exists \xi \varphi$), over Y, is $\Pi_{\pi_Y} \varphi$ (respectively $\Sigma_{\pi_Y} \varphi$), where $\pi_Y \colon X \times Y \to Y$ is straightforward. We must check that the introduction and elimination rules fo and $\forall$ are satisfied: for example, ($\exists$ E) becomes the assertation that given φ o $X \times Y$, φ' over Y, for types X, Y, and a proof (derivation) $\varphi \to \pi_Y^* \varphi'$, there is a pr (derivation) $\Sigma_{\pi_Y} \varphi \to \varphi'$. This is obviously so by adjointness. Also, ($\exists$ I) asserts existence of a proof (derivation) $\varphi \to \pi_Y^* \Sigma_{\pi_Y} \varphi$, which again exists by adjointn (Actually we should allow for a term $t \colon = \langle t', Y \rangle \colon Z \times Y \to X \times Y$. Applying to the above gives the required $t^*\varphi \to \pi_Y'^* \Sigma_{\pi_Y} \varphi$, where now $\pi_Y' \colon Z \times Y \to Y$.) handled similarly.

The equality rules follow as immediate consequences of our definition of E_X $\Sigma_{\Delta_X} \top_X$: (R) asserts the existence of a morphism $\top_X \to \Delta_X^* \Sigma_{\Delta_X} \top_X$, which is the of the adjunction $\Sigma_{\Delta_X} \dashv \Delta_X^*$ at $\top_X$. For (sub), note that for any $\varphi \in |\boldsymbol{P}(X)|$, ther a morphism $s \colon \pi_2^* \varphi \wedge \Sigma_{\Delta_X} \top_X \to \pi_1^* \varphi$: essentially this is just the counit ε for the junction $\Sigma_{\Delta_X} \dashv \Delta_X^*$, by Frobenius Reciprocity. (Consider

$$\pi_2^* \varphi \wedge \Sigma_{\Delta_X} \top_X \xrightarrow{\sim} \Sigma_{\Delta_X} (\Delta_X^* \pi_2^* \varphi \wedge \top_X) \xrightarrow{\sim} \Sigma_{\Delta_X} \varphi \xrightarrow{\sim} \Sigma_{\Delta_X} \Delta_X^* \pi_1^* \varphi \xrightarrow[\varepsilon_{\pi_1^* \varphi}]{} \pi_1^* \varphi .)$$

The actual form of (sub) can be easily derived from this. Note first that b theorem of LAWVERE [10], $\Sigma_{\Delta_{X \times Y}} \top_{X \times Y} \cong \pi_{13}^* \Sigma_{\Delta_X} \top_X \wedge \pi_{24}^* \Sigma_{\Delta_Y} \top_Y$, for project $X \times X \xleftarrow[\pi_{13}]{} X \times Y \times X \times Y \xrightarrow[\pi_{24}]{} Y \times Y$. We will show how to derive the morph corresponding to

$$\dfrac{s = s' \quad t = t' \quad \varphi(s, t)}{\varphi(s', t')}$$

for φ over $X \times Y$, and terms $s, s' \colon X' \to X$, $t, t' \colon Y' \to Y$. This morphism is

$$\pi_{12}'^* (s \times t)^* \varphi \wedge \pi_{13}'^* (s \times s')^* E_X \wedge \pi_{24}'^* (t \times t')^* E_Y \to \pi_{34}'^* (s' \times t')^* \varphi;$$

it is obtained by applying $(s \times t \times s' \times t')^*$ to

$$\pi_{12}^* \varphi \wedge \pi_{13}^* E_X \wedge \pi_{24}^* E_Y \to \pi_{34}^* \varphi,$$

which exists because $\pi_{12}^* \varphi \wedge E_{X \times Y} \to \pi_{34}^* \varphi$ does, by LAWVERE's result.