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HYPERDOCTRINES, NATURAL DEDUCTION AND THE BECK CONDITION

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0. Introduction

In the late sixties F. W. LAWVERE showed that the logical connectives and quantifiers were examples of the categorical notion of adjointness. In [9] and [10] he amplified this notion by a more thorough discussion of the structure of a hyperdoctrine, which had much of the flavour of intuitionistic logic with equality. In this context it was natural to "stratify" formulae and proofs according to the free variables occurring in them, a procedure later to become standard in categorical logic. (See MAKKAI-REYES [12], FOURMAN [1], KOCK-REYES [7], for example.) In this paper, we make the relationship between hyperdoctrines and logic precise, showing that hyperdoctrines are naturally equivalent to first order intuitionistic theories with equality, where here "theory" is intended to include some proof theoretic structure, and not merely the notion of entailment. Moreover, we will show that this equivalence restricts to one giving a natural logical interpretation to the BECK (or CHEVALLEY) condition: in a given hyperdoctrine, the Beck condition for a pullback diagram is just the condition that the corresponding theory "recognizes" the pull back.

This work has an obvious relationship to LAMBEK [8], and to SZABO [21], but, apart from the evident difference in using natural deduction rather than the sequent calculus, one important variant must be noted. SZABO treats the quantifiers as infinite conjunctions and disjunctions, whereas here (following LAWVERE [9]) they are operations adjoint to substitution. This avoids any need to refer to infinitary logic, and more closely reflects their nature: the adjunctions are explicit in the rules for the quantifiers (in either Gentzen system).

There are also connections between hyperdoctrines and Dialectia interpretations (see P. SCOTT [17]) and realizability (see HYLAND, JOHNSTONE, PITTS [5]; note: a "tripos" is a po-hyperdoctrine with a generic predicate). We plan to explore these connections further in a sequel, particularly with respect to GIRARD's type theory (GIRARD [3]).

Basics of category theory may be found in MAC LANE [11] or GOLDBLATT [4].

1. First Order Logic

We base our logic, LPCE, on a natural deduction formulation of intuitionistic, multi-sorted, first order predicate calculus with equality. The main modification we must introduce (essentially to be able to allow interpretations with uninhabited sorts) is

[1]) These results are contained in the author's Ph. D. thesis, SEELY [18].