k, l as shown in figure 1.

(Note that K and L are strict in their first coordinate, lax only in the second.)

$$\begin{tikzcd}
\underline{B}(FA,B) \arrow[r, "K_{AB}"] \arrow[d, "\underline{B}(Fa,B)"'] & \underline{A}(A,GB) \arrow[r, "L_{AB}"] \arrow[d, "\underline{A}(a,GB)"] \arrow[dl, phantom, "k_{aB}"] & \underline{B}(FA,B) \arrow[d, "\underline{B}(Fa,B)"] \arrow[dl, phantom, "l_{aB}"] \\
\underline{B}(FA_1,B) \arrow[r, "K_{A_1B}"] \arrow[d, "\underline{B}(FA_1,b)"'] & \underline{A}(A_1,GB) \arrow[r, "L_{A_1B}"] \arrow[d, "\underline{A}(a_1,Gb)"] \arrow[dl, phantom, "\Leftarrow k_{A_1b}"] & \underline{B}(FA_1,B) \arrow[d, "\underline{B}(FA_1,b)"] \arrow[dl, phantom, "l_{A_1b}\Rightarrow"] \\
\underline{B}(FA_1,B_1) \arrow[r, "K_{A_1B_1}"'] & \underline{A}(A_1,GB_1) \arrow[r, "L_{A_1B_1}"'] & \underline{B}(FA_1,B_2)
\end{tikzcd}$$

Figure 1

(ii) $(L \dashv K)$ There are (lax) modifications

$$\eta : \mathrm{id}(\underline{A}(-, G-)) \Longrightarrow KL$$
$$\epsilon : LK \Longrightarrow \mathrm{id}\ (\underline{B}(F-, -))$$

so that $(K\epsilon)\cdot(\eta K) = K$ and $(\epsilon L)\cdot(L\eta) = L$.

(Again, coherence is relegated to the appendix, where also the meaning of "lax modification" is given.)

3.4 Example: Fix a type E. If G: LAMBDA $\longrightarrow$ LAMBDA is the lax functor $(E \Rightarrow A)$ of 3.2, and if F: LAMBDA $\longrightarrow$ LAMBDA is the (strict) functor $F(A) = (A \,\&\, E)$, then $F \dashv G$.

It is a pity, but setting out this structure in full would take much too much space here; a summary of the relevant ingredients is given in Table 1. As an example, consider η and ϵ: Given objects (types) A,B and a morphism (term) $c: A \longrightarrow (E \Rightarrow B)$,

$$K_{AB}L_{AB}(c) = \lambda\ z \text{ in } E.\ c[x_A := 1st\langle x,z\rangle](2nd\langle x,z\rangle)$$
$$= \lambda\ z \text{ in } E.\ c(z).$$

So $\eta_{AB}(c)$ is the eta conversion

$$c \Longrightarrow \lambda\ z \text{ in } E.\ c(z).$$

Similarly, for a term $d: (A \,\&\, E) \longrightarrow B$, with free variable $\langle x_A, z_E\rangle$, $\epsilon_{AB}(d)$ is the beta conversion $(\lambda\ z \text{ in } E.\ d)\ (z) \Longrightarrow d$.

The "triangle identities" are the following principle, which may be viewed as an analogue to beta conversion at the level of reductions:

(BETA) An eta conversion of an occurrence of a logical symbol followed by beta conversion of the same occurrence is an identity operation, (provided the composite is an "endo-operation", so this makes sense.)

F:	$(-\ \&\ E)$	k_{aB}:	$(\&$ beta$)$
G:	$(E \Rightarrow -)$	k_{Ab}:	$(\Rightarrow$ beta$)$
ι_G:	$(\Rightarrow$ beta$)$	l_{aB}:	$(\&$ beta$)$
τ_G:	$(\Rightarrow$ eta$)$	l_{Ab}:	$(\Rightarrow$ beta$)$
K:	$(\&I,\ \Rightarrow I)$	η:	$(\Rightarrow$ eta$)$
L:	$(\&E,\ \Rightarrow E)$	ϵ:	$(\Rightarrow$ beta$)$

TABLE 1

3.5 Usually in defining a notion of adjunction, it is expected that equivalent "semantic" and "syntactic" formulations exist, (the former being in terms of hom-sets, the latter of units and counits.) For lax adjunctions the situation is rather more complicated, and depends on the precise details of the notion of "laxity". In particular, for the notion of 3.3, although no doubt one could "fudge" an equivalent syntactic formulation, what is striking is that the natural such formulation fails. (Section 4 gives a variant - see SEELY [1977] for a discussion of this case.) Further, it is curious to note the role eta conversion plays: for if we reverse the sense of $(\Rightarrow$ eta$)$, then although LAMBDA is no longer an example of 3.3, it gives nevertheless an example of a natural notion of lax syntactic adjunction (which has no natural semantic equivalent.)

4. Reversing eta

4.1 In this section, we briefly consider the situation when eta conversion is increasing. The first remark, of course, is that we must alter the definition (3.1) of lax functor by reversing ι ; τ remains the same.

(It must be admitted that from a 2-categorical viewpoint, this is highly unsatisfactory, in that ι and τ are now going in reversed senses. Indeed, on this observation could be based a fairly convincing argument that this illustrates just why eta ought not to be increasing.)

However, nevertheless, we can give a neat description of the adjoint structure enjoyed by $\Rightarrow$ in this context as well.

4.2 Definition: Given two lax functors (as in 4.1) $F: \underline{A} \longrightarrow \underline{B}$, $G: \underline{B} \longrightarrow \underline{A}$, by a lax syntactic adjunction $F \dashv\!\!\| G$, we mean that there are lax 2-natural transformations

$$\alpha: \mathrm{id}(\underline{A}) \longrightarrow GF \quad \text{and} \quad \beta: FG \longrightarrow \mathrm{id}(\underline{B})\ ,$$

lax in the sense that for morphisms $a: A_1 \longrightarrow A$

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