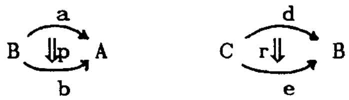
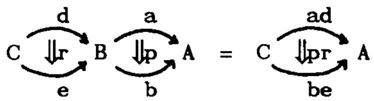


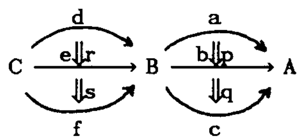
we have a "horizontal composition" giving a 2-cell



This composite is denoted $p{*}r$, or pr if no confusion results. (Here ad and be are given by composition in the category $\underline{A}_o$.)

The composition $*$ must be associative and have as identities the evident identity 2-cells.

Finally, the compositions must be compatible: given



then $(q\cdot p){*}(s\cdot r) = (q{*}s)\cdot(p{*}r)$, (this is known as the "interchange law".) (If we view a 2-category as a CAT-enriched category, KELLY [1982], this is part of the "functorality" of composition: $\underline{A}(B,A) \times \underline{A}(C,B) \longrightarrow \underline{A}(C,A)$.)

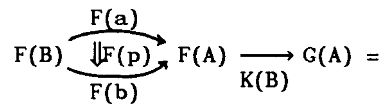
1.2 Examples: The paradigmatic example (for category theorists) is the 2-category CAT of categories, functors, and natural transformations. Indeed, there is an equation (categories: 2-categories) = (SET : CAT), where SET is the (paradigmatic) category of sets and functions.

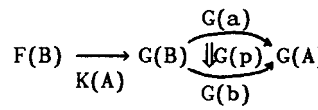
Other examples can be constructed from various categories of ordered objects, where the hom-sets are themselves ordered naturally, and so are categories. For instance, the category $\underline{QD}_o$ of qualitative domains and stable functions (GIRARD [1986]) becomes a 2-category QD by saying there is a 2-cell $f \Longrightarrow g$ just if $f \leq g$ in the Berry order, for stable functions $f,g : X \longrightarrow Y$, (ie. if $f \leq g$ in $X \Rightarrow Y$.)

Finally, as stated in the introduction, the typed lambda calculus may naturally be viewed as a 2-category, as discussed in section 2.

1.3 Definition: A (strict) 2-functor $F : \underline{A} \longrightarrow \underline{B}$ sends objects (respectively morphisms, 2-cells) of $\underline{A}$ to objects (respectively morphisms, 2-cells) of $\underline{B}$, preserving domains, codomains, identities, and compositions.

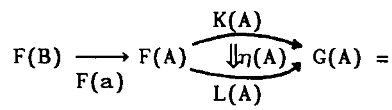
1.4 Similarly, we can define 2-natural transformations. $K:F \Longrightarrow G:\underline{A} \longrightarrow \underline{B}$ assigns to each object A of $\underline{A}$ a morphism $K(A): F(A) \longrightarrow G(A)$ in $\underline{B}$, natural in the usual sense (for $a:B \longrightarrow A$, $K(A)\cdot F(a) = G(a)\cdot K(B)$), and 2-natural, in that for a 2-cell $p:a \Longrightarrow b:B \longrightarrow A$ in $\underline{A}$, we have

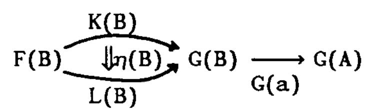




(where this notation means in fact the horizontal composite of, on the left hand side, the identity 2-cell id(K(B)) with F(p), and similarly on the right.) (Note that one frequently identifies an object with its identity map.)

1.5 Definition: A modification $\eta:K \longrightarrow L: F \Longrightarrow G: \underline{A} \longrightarrow \underline{B}$ is a morphism of 2-natural transformations, and assigns to each object A of $\underline{A}$ a 2-cell $\eta(A):K(A) \Longrightarrow L(A)$ so that for $a:B \longrightarrow A$ in $\underline{A}$,





(Again, F(a) means the identity 2-cell, and the equation is between horizontal composite 2-cells.)

(So, 2-CAT is really a 3-category!...)

1.6 In sections 3, 4, we shall also examine weakenings of these notions, in discussing lax functors, weak adjunctions, and so on. For a fuller discussion, see KELLY-STREET [1974], KELLY [1982] and GRAY [1974].

2. The 2-category LAMBDA

2.1 I outlined the structure in the introduction, so now I shall be brief, mainly fixing notation and clarifying some technical points. I assume the reader is familiar with the typed lambda calculus, as in LAMBEK-SCOTT [1986]; the following is a brief summary.

Types are closed under the operations A & B, and $A \Rightarrow B$.

Terms include variables for each type, and are closed under:

Proceedings of the Symposium on Logic in Computer Science, 1987
(Computer Society of the IEEE 1987) 65–71