Topos Theory.

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proofs. Secondly, as a corollary, the reader is never introduced to logical and geometric morphisms, the two types of mappings between topoi. Finally, Goldblatt omits many important developments in categorical logic. For example, any proper text on the subject should certainly include the following topics: internal logic, categorical model theory (particularly the important work of Makkai and Reyes on the model theory of Grothendieck topoi, where we see the important observation that categories are theories and functors are models), and—in light of work mentioned previously—proof theory.

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P. T. JOHNSTONE. ***Topos theory.*** London Mathematical Society monographs, no. 10. Academic Press, London, New York, and San Francisco, 1977, xxiii + 367 pp.

In the past ten years, with the development of the theory of elementary topoi, there has been an increasing use of category theory (and particularly topos theory) in mathematical logic, set theory, model theory, proof theory, computer science, and other fields related to logic (see Johnstone's bibliography). The feature of topos theory that strikes the logician immediately is that it is the first significant and successful formulation of algebraic higher-order logic, a formulation that has recognisable links with algebraic first-order logic via first-order categorical logic, and that clearly shows connections with algebraic infinitary logic; this is essentially the fact that a Grothendieck topos (i.e., infinitary first-order logic) is an elementary topos (i.e., higher-order logic). For example, I will sketch below a recent simple algebraic (i.e., topos-theoretic) proof of the disjunction and explicit definability properties for intuitionistic second-order arithmetic, which shows the power of this algebraic approach. In ***Topos theory,*** Johnstone has produced an extensive and elegant survey of topos theory, which although not primarily addressed to logicians, nevertheless does cover many of the aspects of the subject that are of serious interest to logicians. He uses the language, concepts, and notations of category theory and sheaf theory extensively; and, as he states in his introduction, he quite deliberately does not use the internal language approach until it is necessary. So there is perhaps better propaganda value in some of the standard introductions to categorical logic, such as ***First order categorical logic*** by M. Makkai and G. E. Reyes (Lecture notes in mathematics, vol. 661, Springer-Verlag, 1977) and the articles by A. Kock and G. E. Reyes and by M. P. Fourman in the ***Handbook of mathematical logic*** (North-Holland Publishing Company, 1978, © 1977, pp. 283–313 and 1053–1090). Once convinced of the value of topos theory to logic, however, one could not ask for a clearer guide to the significant aspects of the subject's development than Johnstone's book.

The basic idea of categorical logic is that a logical theory has an intrinsic existence independent of its presentation, and that this existence is best represented by a category; briefly, formulae are represented by their extensions. So a theory *is* a category, and conversely a category *is* a theory. (Actually, which theory it is depends on exactly what structure of the category we are interested in: for example, a Grothendieck topos is both an infinitary first-order theory, and a higher-order type theory.) An interpretation of theories is a functor between them, and a model (in some category $\mathscr{E}$) is a functor from the theory to $\mathscr{E}$, said functors preserving appropriate structure, of course (see the Kock and Reyes article). An elementary topos, from this point of view, is just a higher-order type theory (see the Fourman article). More formally, it is a category $\mathscr{E}$ that has finite limits, is Cartesian closed, and has a subobject classifier Ω. (Cartesian closedness internalises "function types," and Ω internalises propositions—i.e., morphisms $1 \to \Omega$ correspond to subobjects of 1, which correspond to propositions, or formulae with no free variables; more generally, subobjects $X' \rightarrowtail X$ are classified by their "characteristic morphism" $X \to \Omega$.) Two kinds of maps between topoi are relevant: those corresponding to the higher-order logic are logical functors (functors preserving all the topos structure); those corresponding in a sense to some of the first-order logic (to algebraic and geometric aspects of topoi) are geometric morphisms $p = (p_*, p^*) : \mathscr{F} \to \mathscr{E}$, where p_*, p^* are functors $\mathscr{F} \to \mathscr{E}$, $\mathscr{E} \to \mathscr{F}$, and p^* (the *inverse image*) is a left exact left adjoint to p_* (the *direct image*).

Mathematically, topos theory has two major aspects that have been heavily exploited. As a higher-order type theory, a topos shares many properties with models of set theory, so one can internalise much of mathematics to carry it out in an arbitrary topos (just as when working in a Boolean-valued model of set theory). This aspect is captured by the slogan "topoi are universes of continuously variable sets," ("continously variable" from the Grothendieck school of algebraic geometry, for whom a topos was a category of sheaves on a site). The second aspect is that of varying the universe in which one works. Consider, for example, field theory in an arbitrary topos $\mathscr{E}$. Clearly not every field F (in arbitrary $\mathscr{E}$) has an algebraic closure in $\mathscr{E}$, since $\mathscr{E}$ need not satisfy the axiom of choice. If one changes topos, however, an algebraic closure is possible—i.e., there is an $\mathscr{E}$-topos $\mathscr{F}$ (a topos $\mathscr{F}$ with a geometric morphism $p : \mathscr{F} \to \mathscr{E}$) such that p^*F has an algebraic closure in $\mathscr{F}$. So the correct extension of the concept "field extension of F" is not field extensions in $\mathscr{E}$, but field extensions in any $\mathscr{E}$-topos $\mathscr{F}$. This idea of working over an arbitrary base topos $\mathscr{E}$, and combining internal and external aspects of topoi, is central to the topos viewpoint, to applications of topos theory, and to categorical logic generally. A significant application is to the subject of classifying topoi (see the Kock and Reyes article).

After presenting the categorical and topos-theoretic basics in Chapters 0 and 1, Johnstone introduces the internal aspect of topoi in Chapters 2 and 3, dealing with internal category theory, topologies, and sheaves. This also shows the connections with Grothendieck topos theory: not only is a Grothendieck topos an elementary topos, but more importantly, much of Grothendieck topos theory can be generalised to the context of elementary topoi. One example is well known: given a partially ordered set P of forcing conditions, Sets^P is a topos; $\neg\neg$ is a topology (in any topos), and *double negation sheaves*, $\mathrm{sh}_{\neg\neg}(\mathrm{Sets}^P)$, is equivalent to the usual Boolean-valued model (considered as a category) constructed from P, with the Boolean algebra the global elements of $\Omega_{\neg\neg}$, i.e., the algebra $\mathrm{RO}(P)$, the regular open sets of P.

Chapter 4 develops the theory necessary for working over an arbitrary base topos: the factorisation theorem for geometric morphisms, the gluing construction, Diaconescu's theorem, and bounded morphisms. Diaconescu's theorem is central to classifying topoi; the gluing construction has recently been used by P. Freyd to give a strikingly simple proof of the disjunction and explicit definability properties for intuitionistic higher order arithmetic (say for the system HAS, although it is clear the proof has considerable generality). These properties are equivalent to having the terminal object 1 be indecomposable and projective in the free topos-with-natural-numbers-object, $\mathscr{E}_0$ (see the Fourman article for a description of $\mathscr{E}_0$). We glue Γ: $\mathscr{E}_0 \to \mathrm{Sets}$, where $\Gamma(X) = \mathrm{Hom}(1, X)$, to get $\mathscr{E}_1 = \mathrm{Gl}(\Gamma, \mathrm{id})$, a topos with a natural numbers object. (This is the algebraic version of adding new constants $C_{B,V}$, as in ***Metamathematical investigation of intuitionistic arithmetic and analysis***, edited by A. S. Troelstra, Springer-Verlag, 1973.) Trivially, 1 is indecomposable and projective in $\mathscr{E}_1$ (since it is in Sets), and since $\mathscr{E}_0$ is free, the logical functor $\mathscr{E}_0 \to \mathscr{E}_1$ and the canonical projection $\mathscr{E}_1 \to \mathscr{E}_0$ show $\mathscr{E}_0$ is a retract of $\mathscr{E}_1$, and hence 1 must be indecomposable and projective in $\mathscr{E}_0$ also. (Similarly, the only closed terms of type N, $1 \to N$ in $\mathscr{E}_0$ are the numerals.) The only work in this proof is in showing $\mathscr{E}_1$ is a topos with natural numbers object, with structure canonically lifted from $\mathscr{E}_0$, Sets. But this is all part of the general theory. (A similar result for $\mathscr{E}_0 =$ the free Heyting algebra avoids even this; $\mathscr{E}_1$ then is just $\mathscr{E}_0$ with a new maximal element.)

Chapter 5 covers some of the logical aspects of topoi, such as axioms of Booleanness and choice; the internal logic of topoi and forcing ("Kripke–Joyal semantics") are described. Chapter 6 covers the axiom of infinity ("a natural numbers object"), and algebraic and geometric theories, including discussion of classifying topoi (topoi with a "generic" model of the theory; see the Makkai and Reyes book or the Kock and Reyes article for further discussion). Both the elegant categorical construction and the more useful syntactic construction of classifying topoi are described. In Chapter 7, categorical analogues to the completeness theorem for first-order finitary (Deligne's theorem) and infinitary (Barr's theorem) logic are given (the latter being Boolean completeness, of course). For example, Deligne's theorem, that a coherent topos $\mathscr{E}$ has enough points (geometric morphisms p : Sets $\to \mathscr{E}$, or equivalently, models p^* of $\mathscr{E}$), is essentially the fact that first-order finitary coherent logic (see the Kock and Reyes article) has "enough" models, i.e., completeness holds. Chapter 8 covers cohomology, and Chapter 9 deals with aspects of set theory in the topos context: Kuratowski-finiteness, the Cole–Mitchell–Osius equivalence of the consistency of set theory and of topos theory, and Tierney's "toposophical" rendering of Cohen's proof of the independence of the continuum hypothesis. This last, although the details are still essentially set theoretical, does show the conceptual power of topoi in the straightforward construction of the Boolean-valued model as $\mathrm{sh}_{\neg\neg}(\mathrm{Sets}^P)$. An appendix covers locally internal (indexed) categories (see ***Indexed categories and their applications,*** edited by Johnstone and R. Paré, Springer-Verlag, 1978).

Topos theory contains a great deal of material, much of relevance to logicians. I have participated in a graduate course based on this book: the students, from a variety of mathematical backgrounds, found the text clear, though in some of the harder sections other references were helpful in filling in details. There are many exercises to supplement the material. In short, this is an excellent book; I strongly recommend it to readers of this JOURNAL.

There are a number of typographical errors, most not likely to cause confusion. These might be mentioned: In Definition 6.53, the displayed morphism M_α should not have a "tail"—it is not necessarily a monomorphism. In Theorem 7.51, it is $\mathscr{F}$, not $\mathscr{E}$, that should be asserted to satisfy (SG). A more serious error is that the proof of Proposition 7.37 (i) is quite wrong. (This was pointed out by Kathy Edwards.) A correct direct proof can be reconstructed in the fashion of SGA4, *exposé* VI, Exercise 3.11 (p. 232) of [GV] (see Johnstone's bibliography); there is a completely different proof in the Makkai and Reyes book, Chapter 9, §2, which uses logical techniques to construct a pretopos $\mathscr{P}(T_C)$ from a site C for $\mathscr{E}$, which is then used to show that $\mathscr{E}_{\text{coh}}$ is itself a pretopos. The bibliographic reference [MR] should be to the Makkai and Reyes book already cited in this review. And finally, it seems to me that "topoi" is as defensible as "formulae" (see 5.41 (iii) and elsewhere) and that perhaps the only solution to what to do on a cold day (see p. xx) is for each to carry his own thermos. (Review submitted in 1979.) ROBERT SEELY

WALTER TAYLOR. ***Equational logic.*** Houston journal of mathematics, survey 1979. Department of Mathematics, University of Houston, Houston 1979, iii + 83 pp.

Most "universal algebraists" would agree to a definition of universal algebra as the "model theory of equations," but none of the books about universal algebra seems to reflect this aspect of the subject. W. Taylor's monograph ***Equational logic*** can be regarded as a survey over the whole of universal algebra from the viewpoint of equations.

This work surveys the role of equations in the theory of general or classical algebras. It contains no proofs but has many illustrative examples, many open problems, and over eight hundred references. §§1–7 review the necessary facts about basic notions such as free algebras, equations, varieties, subdirect products, subdirectly irreducible and simple algebras, equational theories, and equivalence of varieties. §8 contains examples of algebras generic for certain varieties and §9 deals with the question of when an equational theory is finitely based. Again, many examples and counterexamples are given, such as a finite ring with no finite equational base. §10 specializes to 1-based theories and §11 deals with irredundant bases. §12 surveys various decidability questions such as word problems and decision problems for finite algebras or finite sets of equations, again with many examples and problems. §13 investigates the lattice of equational theories, illustrating different possibilities by various examples. In §14 several invariants of varieties are studied, such as spectra, categoricity, varietal chains, size of free algebras, amalgamation, congruence extension, residual smallness, and others. §15 gives a survey concerning Malcev-conditions and §16 deals with the impact of the topological structure of an algebra on its equations. Finally, §17 lists some additional topics that do not fit into any of the previous sections. HEINRICH WERNER

GEORGE GRÄTZER. ***Universal algebra.*** Second edition, with new appendices and additional bibliography, of XXXVIII 643. Springer-Verlag, New York, Heidelberg, and Berlin, 1979, xviii + 581 pp.

GEORGE GRÄTZER. *Appendix 1. General survey.* Therein, pp. 331–34.

GEORGE GRÄTZER. *Appendix 2. The problems.* Therein, pp. 342–347.

Since the first day of its appearance in 1968 this book has been *the* standard reference in universal algebra, and no book since has reached its quality in combining an introductory text for the beginner with a comprehensive survey for the researcher. In the new edition the author has left the main text unchanged and has extended it by seven appendices and a large additional bibliography. These take account of the new developments in the field. Like the first edition, this book covers neither systematic treatments of infinitary algebras nor the category-theoretic approach to universal algebra. For a review of the first part of this book the reader is referred to K. Baker's review of the first edition in this JOURNAL (XXXVIII 643). Here only the material added in the second edition will be reviewed.

Appendix 1 (*General survey*) and Appendix 2 (*The problems*) give references to new developments in topics covered by the first edition and to solutions or partial solutions of the problems.