Ignore the small stuff

When calculating limits (as $x \to \infty$) of rational functions, you are taught to divide top and bottom of the fraction by the highest power, thus:

Example: Calculate $\displaystyle\lim_{x\to\infty} \frac{5x^3 - 7x^2 + 1}{4x - x^2 - 7x^3}$.

Answer: Divide top and bottom by x^3:

$$\begin{aligned}
\lim_{x\to\infty} \frac{5x^3 - 7x^2 + 1}{4x - x^2 - 7x^3} &= \lim_{x\to\infty} \frac{5 - \frac{7}{x} + \frac{1}{x^3}}{\frac{4}{x^2} - \frac{1}{x} - 7} \\
&= \frac{5 - 0 + 0}{0 - 0 - 7} \\
&= -\frac{5}{7}
\end{aligned}$$

But this process has the effect of "wiping out" all terms on top and on the bottom whose power is less than the highest power, so that in effect you are calculating the limit thus:

$$\begin{aligned}
\lim_{x\to\infty} \frac{5x^3 - 7x^2 + 1}{4x - x^2 - 7x^3} &= \lim_{x\to\infty} \frac{5x^3}{-7x^3} \\
&= \lim_{x\to\infty} \frac{5}{-7} \\
&= -\frac{5}{7}
\end{aligned}$$

And this is in fact exactly what you are doing:

Theorem: Given functions $f(x), g(x), h(x), k(x)$ with the properties that

$$\lim_{x\to a} \frac{h(x)}{f(x)} = 0 \quad \text{and} \quad \lim_{x\to a} \frac{k(x)}{g(x)} = 0$$

then

$$\lim_{x\to a} \frac{f(x) + h(x)}{g(x) + k(x)} = \lim_{x\to a} \frac{f(x)}{g(x)}$$

Proof: This is obvious if you consider that $f + h = f(1 + h/f)$ and $g + k = g(1 + k/g)$. Note that this also is valid for the cases when a is $\pm\infty$.

In other words, you can "ignore the small stuff". It's worth noticing that this works for horizontal asymptotes for any rational function, since $\displaystyle\lim_{x\to\infty} \frac{x^k}{x^n} = 0$ whenever $k < n$.

I'll leave it to you to extend this to simple algebraic functions, particularly to fractions including roots, and to differences of square roots. Try the following exercises.

Exercises: Calculate these limits:

$$\lim_{x\to\infty} \frac{5 - 3x + 2x^3}{x^3 - 7x^2 + 100};\ \lim_{x\to\infty} \frac{5 - 3x + 2x^3}{x^4 - 7x^2 + 100};\ \lim_{x\to\infty} \frac{5 - 3x + 2x^4}{x^3 - 7x^2 + 100};$$

$$\lim_{x\to\infty} \frac{\sqrt{5 - 3x + 2x^6}}{x^3 - 7x^2 + 100};\ \lim_{x\to-\infty} \frac{\sqrt{5 - 3x + 2x^6}}{x^3 - 7x^2 + 100};\ \lim_{x\to\infty} \left(\sqrt{x^2 + 4x - 6} - x\right)$$

Answers: $2, 0, \infty, \sqrt{2}, -\sqrt{2}, 2$