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Test 2
(version A)

Calculus III (Maths 201–DDB)

(Marks)

Note: Justify all your answers — don't make me guess your thoughts!

(3×2) 1. Let Λ be the line passing through the point $P_0(1,2,3)$, parallel to the vector $\boldsymbol{v} = 3\boldsymbol{i} + 9\boldsymbol{j} + \boldsymbol{k}$, and let Π be the plane given by $2x - y + 3z = 5$.

- (a) Show that Λ is parallel to the plane Π.
- (b) What is the distance from Λ to Π?

  (Hint: Choose a point Q on Π, then project the vector $\overrightarrow{P_0Q}$ onto the normal of the plane Π. You know the length of the projection, right?)
- (c) What is the angle between Π and the plane $3x + 2y + z = 1$?

(2) 2. Show that for any vector function $\boldsymbol{r}(t)$ which is parallel to $\boldsymbol{r}''(t)$ (for all t), the cross product $\boldsymbol{r}(t) \times \boldsymbol{r}'(t)$ is constant.

(3) 3. $\boldsymbol{r} = \boldsymbol{r}(t)$ is a vector function: simplify $\frac{d}{dt}(\boldsymbol{r} \cdot (\boldsymbol{r}' \times \boldsymbol{r}''))$ as much as possible. Is this derivative a vector function or a scalar function?

(Hint: for questions 2,3 don't use components; instead use the general properties of derivatives and dot and cross products of vectors.)

(9) 4. Name and sketch the following surfaces in 3-space. Show all your work, including traces, intercepts (and contour curves if you use them).
(a) $\rho = 4\sec\phi$ (b) $z = x^2 - y^2$ (c) $z - r^2 = 4$

(5) 5. Find the parametrization of $\boldsymbol{r}(t) = \langle 3\cos t\,,\, 3\sin t\,,\, 4\rangle$ in terms of the arclength s. Use $t = 0$ for the "starting point". This curve lies on a quadric surface: identify such a surface (give its equation) and draw a sketch of the graph of the curve and the surface on which it lies.

(12) 6. A particle P moves along a curve $\boldsymbol{r}(t) = 2t\,\boldsymbol{i} + \ln t\,\boldsymbol{j} + t^2\,\boldsymbol{k}$, $t > 0$. Find:
(a) the unit tangent vector $\boldsymbol{T}(t)$; (b) the unit normal vector $\boldsymbol{N}(t)$; (c) the curvature $\kappa(t)$;
(d) the tangential and normal components a_T, a_N of acceleration;
(e) the length of the part of the curve from $t = 1$ to $t = 2$.

(Hint: do the algebra ***carefully****, and you will find there is a lot of simplification—all the square roots work out easily if you factor where appropriate.)*

(5) 7. Find the point (x, y) of maximum curvature for the parabola $y = 1 - \frac{1}{2}x^2$. Find the equation of the osculating circle at that point. Draw the graphs of the parabola and the osculating circle (on the same axes).

(4) 8. Is the following function continuous at the origin? (Justify your answer.)

$$f(x,y) = \begin{cases} \dfrac{x^2 - y^2}{x^2 + y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

(4) 9. Show that $z = \sqrt{\dfrac{x}{y}}$ is a solution of the equation $\dfrac{x}{z}\dfrac{\partial z}{\partial x} + \dfrac{y}{z}\dfrac{\partial z}{\partial y} = 0$.

(Total: 50)