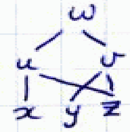
Cal III Test 3 Solutions



① a) $\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{3x^2y+z^3}{y^3+3z^2x}$

b) $\frac{\partial w}{\partial x} = z f_u$; $\frac{\partial w}{\partial y} = z f_v$; $\frac{\partial w}{\partial z} = x f_u + y f_v$, so (qed)

② (a) $\vec{u} = -\frac{\langle 2,1,1\rangle}{\sqrt{6}}$, $T_{\vec{u}} = \vec{\nabla}T\cdot\vec{u} = -\frac{18}{49\sqrt{6}}$

$\vec{\nabla}T = \left\langle \frac{yz(1-x^2+y^2+z^2)}{(1+x^2+y^2+z^2)^2}, \frac{xz(1+x^2-y^2+z^2)}{(1+x^2+y^2+z^2)^2}, \frac{xy(1+x^2+y^2-z^2)}{(1+x^2+y^2+z^2)^2} \right\rangle$

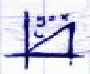
at P_0 : $= \left\langle -\frac{1}{49}, \frac{10}{49}, \frac{10}{49} \right\rangle$

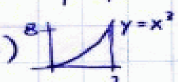
(b) $\vec{\nabla}T = \langle -\frac{1}{49}, \frac{10}{49}, \frac{10}{49}\rangle$ (c) $|\vec{\nabla}T| = \frac{\sqrt{201}}{49}$

③ $\vec{\nabla}f = \langle y - \frac{1}{x^2}, x + \frac{64}{y^2}\rangle = \vec{0}$ if ($x^2 = \frac{1}{y}$ so, since $x \neq 0 \neq y$, $x = -\frac{1}{4}, y = 16$)

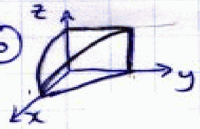
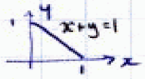
$D = \begin{vmatrix} 2/x^3 & 1 \\ 1 & -128/y^3 \end{vmatrix} = -\left[\frac{256}{x^3y^3}+1\right] = 3 > 0$, $f_{xx} = -128 < 0$, so $(-\frac{1}{4}, 16)$ is a <u>max</u>

④ $f = d^2 = (x-1)^2 + (y-2)^2 + z^2$, $g = x^2+y^2-z^2$, so $\vec{\nabla}f = \lambda\vec{\nabla}g$, $g = 0$ becomes $\{2x-2 = 2\lambda x;\ 2y-4 = 2\lambda y;\ 2z = -2\lambda z;\ z^2 = x^2+y^2\}$. Either $z=0$ or $\lambda = -1$: If $z = 0$, then $x = 0 = y$ also. If $\lambda = -1$, $x = \frac{1}{2}$, $y = 1$, $z = \pm\frac{\sqrt{5}}{2}$.
Comparing f values, we see the minimum distance ($\frac{\sqrt{10}}{2}$) is at the points $(\frac{1}{2}, 1, \pm\frac{\sqrt{5}}{2})$

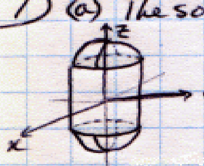
⑤ (a)  $\int_0^1\int_0^x \frac{e^x}{x}\,dy\,dx = \int_0^1 e^x\,dx = e-1$

(b)  $\int_0^2\int_0^{x^3} e^{x^4}\,dy\,dx = \int_0^2 x^3 e^{x^4}\,dx = \frac{1}{4}(e^{16}-1)$

(c)  $\int_0^{\pi/4}\int_0^{2\sqrt{2}} r^2\,dr\,d\theta = \frac{\pi}{4}\cdot\frac{1}{3}(2\sqrt{2})^3 = \frac{4\sqrt{2}\pi}{3}$

⑥   $\int_0^1\int_0^{1-x}(1-x^2)\,dy\,dx = \int_0^1(1-x)(1-x^2)\,dx = x - \frac{x^2}{2} - \frac{x^3}{3} + \frac{x^4}{4}\Big]_0^1 = \frac{5}{12}$

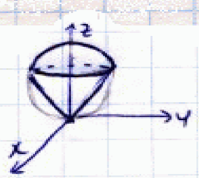
⑦ (a) The solid consists of a cylinder (radius $= \sqrt{2}$) cut out of a sphere (radius $= \sqrt{8}$)



(b) (i) $\int_0^{2\pi}\int_0^{\sqrt{2}}\int_{-\sqrt{8-r^2}}^{\sqrt{8-r^2}} r\,dz\,dr\,d\theta = 2\pi\int_0^{\sqrt{2}} 2r\sqrt{8-r^2}\,dr = 2\pi\left[\frac{32\sqrt{2}}{3} - 4\sqrt{6}\right]$

$= \frac{64\sqrt{2}}{3}\pi - 8\pi\sqrt{6}$

(ii) $\int_0^{2\pi}\int_0^{\frac{\pi}{6}}\int_0^{\sqrt{8}} \rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta + \int_0^{2\pi}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}}\int_0^{\sqrt{2}\csc\varphi} \rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta + \int_0^{2\pi}\int_{\frac{5\pi}{6}}^{\pi}\int_0^{\sqrt{8}} \rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta$

⑧ 

$V = \iiint_S dV = \int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_0^{4\cos\varphi} \rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta$

$= 2\pi\cdot\int_0^{\frac{\pi}{4}} \frac{4^3}{3}\cos^3\varphi\sin\varphi\,d\varphi$

$x^2+y^2+z^2 = 4z$
$\rho^2 = 4\rho\cos\varphi$
$\rho = 4\cos\varphi$

$= 2\pi\cdot\frac{4^3}{3}\cdot\frac{1}{4}\cdot\frac{3}{4} = 8\pi$

⑨ Note $H_x = -\frac{1}{x^2}(zH_u + yH_v)$; $H_y = \frac{1}{x}H_v$; $H_z = \frac{1}{x}H_u$

So $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = x\left(-\frac{H_x}{H_z}\right) + y\left(-\frac{H_y}{H_z}\right) = \left(z + y\frac{H_v}{H_u}\right) - y\frac{H_v}{H_u} = z$.