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Test 3
(version for practice)

## Calculus III (Maths 201–DDB)

1. (a) If $x^3y + y^3z + z^3x = 0$ defines $z = f(x, y)$, find $\dfrac{\partial z}{\partial x}$.
   (b) If $w = f(xz, yz)$ (and $f$ is a differentiable function), show that
   $$x\,\frac{\partial w}{\partial x} + y\,\frac{\partial w}{\partial y} = z\,\frac{\partial w}{\partial z}$$

2. In a certain space, the temperature at a point $P(x, y, z)$ is given by $T = \dfrac{xyz}{1 + x^2 + y^2 + z^2}$.
   (a) Find the directional derivative of $T$ at $P_0(2, 1, 1)$ in the direction of a vector pointing **from** $P_0$ **to** the origin $O$.
   (b) Find the direction of greatest increase in $T$ at the point $P_0$.
   (c) What is the greatest increase in $T$ at $P_0$?

3. Find and classify the critical points of $f(x, y) = \dfrac{1}{x} - \dfrac{64}{y} + xy$.

4. Use Lagrange Multipliers to find the minimum distance between the point $(1, 2, 0)$ and the cone $z^2 = x^2 + y^2$.

5. Evaluate the following: (change coordinates as appropriate; don't worry about the fact that the first one is in fact an improper integral, as that has no effect on the value in this case.)
   (a) $\displaystyle\iint_R \frac{e^x}{x}\,dA$, where $R$ is the triangular region with vertices $(0, 0), (1, 0), (1, 1)$.
   (b) $\displaystyle\int_0^8 \int_{\sqrt[3]{y}}^2 e^{x^4}\,dx\,dy$
   (c) $\displaystyle\int_0^2 \int_y^{\sqrt{8-y^2}} \sqrt{x^2 + y^2}\,dx\,dy$

6. Sketch the solid region $\mathcal{S}$ in the first octant (*i.e.* where $x \geq 0, y \geq 0, z \geq 0$), bounded by the cylinder $z = 1 - x^2$ and the plane $y = 1 - x$. Find the volume of $\mathcal{S}$.

7. The volume of a solid region $\mathcal{S}$ can be found by evaluating the following triple integral
   $$\int_{-\sqrt{2}}^{\sqrt{2}} \int_{-\sqrt{2-x^2}}^{\sqrt{2-x^2}} \int_{-\sqrt{8-x^2-y^2}}^{\sqrt{8-x^2-y^2}} dz\,dy\,dx$$
   (a) Sketch $\mathcal{S}$.
   (b) Express the above integral in both
      i. cylindrical and
      ii. spherical coordinates.

   Evaluate whichever you prefer to find the volume of $\mathcal{S}$.

8. Set up the triple integral needed to find the volume of the ("ice-cream cone") solid region $\mathcal{IC}$ inside the sphere $x^2 + y^2 + z^2 = 4z$ and above the cone $z = \sqrt{x^2 + y^2}$. Calculate the volume of the solid $\mathcal{IC}$.

9. **(Bonus question)** Suppose $H(u, v)$ is differentiable, and that $z = z(x, y)$ is implicitly defined by the equation $H\left(\dfrac{z}{x}, \dfrac{y}{x}\right) = 0$ (for $x \neq 0 \neq z$). Show that $x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y} = z$.