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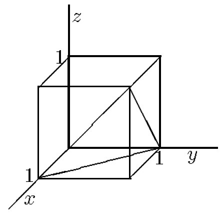
Test 2
(version for practice)

Calculus III (Maths 201–DDB)

Justify all your answers — just having the correct answer is not sufficient.



1. What is the equation of the plane containing the triangle defined by three adjacent diagonals (on the bottom face, on the front face, and on the right face) as shown in the following unit cube?
   What is the angle between this plane and the xy plane?

2. Name and sketch the following surfaces in 3-space. Show all your work, including traces, intercepts (and contour curves if you use them).
   (a) $r^2 - z^2 = 4$ (b) $z = 2y^2 - x^2$ (c) $\rho \sin(\phi) = 1$

3. Find all points (x, y) where the graph of $\mathbf{r}(t) = \sin t\,\mathbf{i} + \sin^2 t\,\mathbf{j}$ is *not* smooth.
   Sketch the graph (in the xy plane) and explain what geometric feature of the graph corresponds to the lack of smoothness at those points.

4. For the parabolic helix $\mathbf{r} = \langle t\cos t, t\sin t, t^2\rangle$:
   (a) What is the equation of the paraboloid on which this curve lies? *(Hint: Pythagoras)*
   (b) Sketch the graphs of the paraboloid and the part of the helix from $t = 0$ to $t = 4\pi$, indicating the direction of increasing t.

5. (a) Given perpendicular unit vectors $\mathbf{a}, \mathbf{b}$ and an arbitrary vector $\mathbf{u} = r\,\mathbf{a} + s\,\mathbf{b}$ in the plane spanned by $\mathbf{a}, \mathbf{b}$, show that the scalar components r, s are given by $r = \mathbf{u}\cdot\mathbf{a}$ and $s = \mathbf{u}\cdot\mathbf{b}$.
   (b) Prove that $\frac{d}{dt}(\mathbf{r}\times\mathbf{r}') = \mathbf{r}\times\mathbf{r}''$.

6. A particle P moves along a curve $\mathbf{r}(t) = \mathrm{e}^t\,\mathbf{i} + \sqrt{2}\,t\,\mathbf{j} + \mathrm{e}^{-t}\,\mathbf{k}$. Find: the unit tangent vector $\mathbf{T}(t)$; the unit normal vector $\mathbf{N}(t)$; and the curvature $\kappa(t)$. Find the tangential and normal components $a_{\mathbf{T}}, a_{\mathbf{N}}$ of acceleration. *(Hint: do the algebra carefully, and you will find there is a lot of simplification—all the square roots work out easily if you factor where appropriate.)*

7. Find the (radius, center, and) equation of the osculating circle for the hyperbola $y = \dfrac{1}{x}$ at the point $(1, 1)$. Draw a graph of the hyperbola and the osculating circle (on the same axes). *(Hint: remember the 1-1-$\sqrt{2}$ right-angled triangle.)*

8. Calculate the following limits (if possible; if the limit does not exist, say so, and justify your answer).
   (a) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^4+y^2}$ (b) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2}$ (c) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{x^4+y^2}$

9. Show that $z = 4\sqrt[3]{x^2y}$ is a solution of the equation $x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y} = z$.

Brief Answers: (1): $x+y-z=1$, $54.7°$ (2): Hyperboloid of 1 sheet; Hyperbolic paraboloid; Circular cylinder (3): Not smooth at "endpoints" $(\pm 1, 1)$ (4): $z = r^2$; 2 revolutions around the paraboloid (5) (use standard equations) (6): $\mathbf{T} = \frac{1}{\mathrm{e}^t+\mathrm{e}^{-t}}\langle \mathrm{e}^t, \sqrt{2}, -\mathrm{e}^{-t}\rangle$; $\mathbf{N} = \frac{1}{\sqrt{2}(\mathrm{e}^t+\mathrm{e}^{-t})}\langle 2, -\sqrt{2}(\mathrm{e}^t-\mathrm{e}^{-t}), 2\rangle$; $\kappa = \frac{\sqrt{2}}{(\mathrm{e}^t+\mathrm{e}^{-t})^2}$; $a_{\mathbf{T}} = \mathrm{e}^t - \mathrm{e}^{-t}$; $a_{\mathbf{N}} = \sqrt{2}$. (7): $(x-2)^2 + (y-2)^2 = 2$ (8) 0; DNE; DNE (9) (direct calculation)
(Graphs on webpage)