Instructor: Dr. R.A.G. Seely — Assignment 4

Calculus III (Maths 201–DDB)

Questions 1 – 11 appeared on Assignment $3\frac{1}{2}$.
I repeat them here again (small print) just "for the record".
I will ask you to hand in only a selection of these questions.

1. Sketch the domain of $f(x,y)=\sqrt{36-4x^2-9y^2}$.
2. Sketch level curves of $f(x,y)=x-y^2$ for $z=-2,-1,0,1,2$. Use these to sketch the surface $z=f(x,y)$.
3. Sketch the level surface of $f(x,y,z)=x^2+y^2-z^2$ for $w=1$.
4. Calculate these limits.
   (a) $\displaystyle\lim_{(x,y)\to(0,0)}\left(1+\tan\left(\frac{1}{x+y}\right)\right)$ (b) $\displaystyle\lim_{(x,y)\to(0,0)}\frac{x^2+3y-7}{2+x^3-5y^2}$
   (c) $\displaystyle\lim_{(x,y)\to(0,0)}(1+x^2y^2)^{1/(2x^2y^2)}$
5. Are the following functions continuous at the origin?
   (a) $f(x,y)=\begin{cases}\frac{5xy^2}{2x^2+2y^2} & \text{if } (x,y)\neq(0,0)\\ 0 & \text{if } (x,y)=(0,0)\end{cases}$ (b) $f(x,y)=\begin{cases}\frac{5xy}{2x^2+y^2} & \text{if } (x,y)\neq(0,0)\\ 0 & \text{if } (x,y)=(0,0)\end{cases}$
6. Find $\frac{\partial z}{\partial x}$ for each of the following.
   (a) $z=x^4\cos(xy^3)$ (b) $xyz^2-x^2y=\sin(3z)$ (c) $z=u^2\ln v$ where $u=x^2+y^2, v=xy^3$
7. Demonstrate that $z=\mathrm{e}^{2ax+2y}+\tan^{-1}(ax+y)$ (with a constant) is a solution of the partial differential equation $\frac{\partial z}{\partial x}-a\frac{\partial z}{\partial y}=0$
8. Show that $z=\mathrm{e}^{x-y}\sin(x-y)$ satisfies $\frac{\partial^2 z}{\partial x^2}+2\frac{\partial^2 z}{\partial x\partial y}+\frac{\partial^2 z}{\partial y^2}=0$.
9. If $w=\frac{xy}{x^2+y^2}$, find $\frac{\partial w}{\partial x}$, $\frac{\partial w}{\partial y}$ and then show $x^2\frac{\partial^2 w}{\partial x^2}+2xy\frac{\partial^2 w}{\partial x\partial y}+y^2\frac{\partial^2 w}{\partial y^2}=0$
10. Show that Laplace's equation $\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}=0$ is satisfied by each of the following functions:
    (a) $z=x^3-3xy^2$ (b) $z=e^{-y}\cos x$ (c) $z=\tan^{-1}\left(\frac{y}{x}\right)$ where $(x,y)\neq(0,0)$.
11. (a) Let $f(x,y)=\frac{2xy}{x^2+y^2}$ $(x,y)\neq(0,0)$, $f(0,0)=0$.
    i. find $f_x(0,0)$, $f_y(0,0)$; ii. find both $f_x(x,y)$ and $f_y(x,y)$ at $(x,y)\neq(0,0)$
    (b) Although both the partial derivatives $f_x(0,0)$ and $f_y(0,0)$ exist, show that $f(x,y)$ is not continuous at $(0,0)$.
    (c) Similarly, show that $f_x(x,y)$ and $f_y(x,y)$ are not continuous at $(0,0)$.
12. Question 7: Demonstrate that $z=\mathrm{e}^{2ax+2y}+\tan^{-1}(y+ax)$ (a constant) is a solution of the partial differential equation $\frac{\partial z}{\partial x}-a\frac{\partial z}{\partial y}=0$.

    This is a special case of the more general result: if $w=f(ax+by)$ for a differentiable function f, with a,b constants, then $b\frac{\partial w}{\partial x}-a\frac{\partial w}{\partial y}=0$. Show that this is always true.
13. $w=f(x,y)$ is a continuous function with continuous partial derivatives. Suppose that one substitutes the polar coordinates $x=r\cos\theta$ and $y=r\sin\theta$. Show that $\frac{\partial w}{\partial\theta}=xf_y-yf_x$.

    This is also frequently expressed as $\frac{1}{r}\frac{\partial w}{\partial\theta}=-f_x\sin\theta+f_y\cos\theta$.

    Calculate $\frac{\partial w}{\partial r}$, and show that $\left(\frac{\partial w}{\partial x}\right)^2+\left(\frac{\partial w}{\partial y}\right)^2=\left(\frac{\partial w}{\partial r}\right)^2+\frac{1}{r^2}\left(\frac{\partial w}{\partial\theta}\right)^2$.
14. Show that if f,g have continuous second derivatives then $z(x,t)=f(x+at)-g(x-at)$ is a solution of the wave equation $\frac{\partial^2 z}{\partial t^2}=a^2\frac{\partial^2 z}{\partial x^2}$.
15. Suppose $f(u,v)$ is a differentiable function of two variables, and $w=xyf(xz,yz)$. Show that $x\frac{\partial w}{\partial x}+y\frac{\partial w}{\partial y}-z\frac{\partial w}{\partial z}=2w$.
16. Show that the space curve given by $\boldsymbol{r}(t)=\langle\ln t, t^2-2, t\rangle$ is tangent to the surface given by $xz^2-yz+\cos(xy)=2$ at the point $P(0,-1,1)$.
17. Find the tangent plane to the surface $xyz=27$ at the point $(1,9,3)$.

18. Find the point where the tangent plane to the surface $z = x^2 + 2xy + 2y^2 - 6x + 8y$ is horizontal.

19. The surfaces $\frac{1}{a}x^2 + y^2 - z = a$ and $x^2 + y^2 + z = a^2$ (two paraboloids) share the same tangent plane at the point $P(a, 0, 0)$. Find the value of the constant a.

20. Let $\mathcal{C}$ be the cross section of the hyperboloid $z^2 = x^2 + 4y^2 - 1$ on the plane $x - y + 2z = 6$ (*i.e.* the intersection of the plane and the hyperboloid). Find the equations of the line tangent to $\mathcal{C}$ at the point $P_0(1, -1, 2)$. Hint: find two non-parallel vectors perpendicular (normal) to $\mathcal{C}$ at P_0.

21. Find the directional derivative of the function $f(x, y, z) = x^2 + y^2 + xyz$ at the point $P_0(1, 1, -1)$ in the direction $\boldsymbol{i} - 2\,\boldsymbol{j} + 2\,\boldsymbol{k}$. Find the equation of the tangent plane to the surface $f(x, y, z) = 1$ (same f) at the (same) point P_0.

22. In which direction does $f(x, y, z) = \mathrm{e}^{xy} + z^2$ increase most rapidly at $(0, 2, 3)$? At what rate does f change in that direction?

23. If the temperature T at a point $P(x, y, z)$ on a surface is given by $T(x, y, z) = 3x^2 + 2y^2 - 4z$, find the rate of change in T at P in the direction PQ, where Q is the point $Q(-4, 1, -2)$. Find the maximum rate of change in T at P and the direction in which this occurs.

24. Use the directional derivative to estimate how much $f(x, y) = \cos(\pi xy) + xy^2$ will change if one moves from $(-1, 1)$ a distance of 0.1 unit along the vector $\boldsymbol{i} + \boldsymbol{j}$.

25. Find the directional derivative of $f(x, y, z) = x^3 - xy^2 - z$ at the point $P_0(1, 1, 0)$ in the direction of the vector $2\,\boldsymbol{i} - 3\,\boldsymbol{j} + 6\,\boldsymbol{k}$.

26. For a differentiable function $f(x, y)$, at a fixed point P_0, the following is true: the derivative of f in the direction $\boldsymbol{i} + \boldsymbol{j}$ is $2\sqrt{2}$; the derivative of f in the direction $\boldsymbol{i} - \boldsymbol{j}$ is $3\sqrt{2}$. Calculate ∇f at P_0, and find the maximum rate of increase in f at P_0.

27. Find the points where $f(x, y) = x^3 + y^3 - 3x^2 - 3y^2 - 9x$ has a local max, a local min, and a saddle point.

28. Find and classify the critical points of $f(x, y) = x^3 - 3xy + y^3$.

29. Find and classify the critical points of $f(x, y) = x^2 + 2y^2 - x^2 y$.

30. Suppose $f(x, y, z) = x^2 + y^2 - z$. Show that the line $x = 2 + t\ ,\ y = -t\ ,\ z = 2$ is tangent to the surface $f(x, y, z) = 0$. Find the rate of change in f in the direction of the given line (take the direction of increasing t).

31. Use the method of Lagrange Multipliers to find the maximum and the minimum of the function $x + 2y + 4z$ subject to the constraint $x^2 + y^2 + z^2 = 19$.

    For what values of k is the plane $x + 2y + 4z = k$ tangent to the sphere $x^2 + y^2 + z^2 = 19$? Find the points of tangency. (Is there a connection between these questions?)

32. Find the extreme values of $f(x, y) = x^2 + y^2 - 2x - 4y$ on the disk $x^2 + y^2 \le 16$.

33. Find the maximum possible product of three positive numbers whose sum is 120.

34. Find the point(s) on the surface $z = \frac{1}{2}(x^2 + y^2)$ closest to the point $P(0, 4, 1)$.

35. A rectangle is inscribed in the ellipse $4x^2 + 9y^2 = 36$ in such a way that its sides are parallel to the axes. What are the dimensions of such a rectangle of maximum area? What are the dimensions for the rectangle of minimum area?

36. Find the maximum and minimum values of $f(x, y, z) = z - x^2 - y^2$ on the curve given by the intersection of the plane $x + z = 1$ and the cylinder $x^2 + y^2 = 4$.