Calculus III (Maths 201–DDB)

This is a preview of Assignment 4, and is just for practice for the next test.

1. Sketch the domain of $f(x,y) = \sqrt{36 - 4x^2 - 9y^2}$.

2. Sketch level curves of $f(x,y) = x - y^2$ for $z = -2, -1, 0, 1, 2$. Use these to sketch the surface $z = f(x,y)$.

3. Sketch the level surface of $f(x,y,z) = x^2 + y^2 - z^2$ for $w = 1$.

4. Calculate these limits.
   (a) $\lim_{(x,y)\to(0,0)} \left(1 + \tan\left(\frac{1}{x+y}\right)\right)$ (b) $\lim_{(x,y)\to(0,0)} \frac{x^2 + 3y - 7}{2 + x^3 - 5y^2}$
   (c) $\lim_{(x,y)\to(0,0)} (1 + x^2y^2)^{1/(2x^2y^2)}$

5. Are the following functions continuous at the origin?
   (a) $f(x,y) = \begin{cases} \frac{5xy^2}{2x^2+2y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$ (b) $f(x,y) = \begin{cases} \frac{5xy}{2x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$

6. Find $\frac{\partial z}{\partial x}$ for each of the following.
   (a) $z = x^4 \cos(xy^3)$ (b) $xyz^2 - x^2y = \sin(3z)$ (c) $z = u^2 \ln v$ where $u = x^2 + y^2, v = xy^3$

7. Demonstrate that $z = \mathrm{e}^{2ax+2y} + \tan^{-1}(ax + y)$ (with a constant) is a solution of the partial differential equation $\frac{\partial z}{\partial x} - a\frac{\partial z}{\partial y} = 0$

8. Show that $z = \mathrm{e}^{x-y} \sin(x - y)$ satisfies $\frac{\partial^2 z}{\partial x^2} + 2\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = 0$.

9. If $w = \frac{xy}{x^2+y^2}$, find $\frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}$ and then show $x^2\frac{\partial^2 w}{\partial x^2} + 2xy\frac{\partial^2 w}{\partial x \partial y} + y^2\frac{\partial^2 w}{\partial y^2} = 0$

10. Show that Laplace's equation $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$ is satisfied by each of the following functions:
    (a) $z = x^3 - 3xy^2$ (b) $z = e^{-y}\cos x$ (c) $z = \tan^{-1}\left(\frac{y}{x}\right)$ where $(x,y) \neq (0,0)$.

11. (a) Let $f(x,y) = \frac{2xy}{x^2+y^2}$ $(x,y) \neq (0,0)$, $f(0,0) = 0$.
    i. find $f_x(0,0)$, $f_y(0,0)$; ii. find both $f_x(x,y)$ and $f_y(x,y)$ at $(x,y) \neq (0,0)$
    (b) Although both the partial derivatives $f_x(0,0)$ and $f_y(0,0)$ exist, show that $f(x,y)$ is not continuous at $(0,0)$.
    (c) Similarly, show that $f_x(x,y)$ and $f_y(x,y)$ are not continuous at $(0,0)$.