Calculus III (Maths 201–DDB)

You will be asked to hand in only a selection of these problems for grading.

1. Compute the angle between the planes $x = 7$ and $x + y + \sqrt{2}\,z = -3$

2. A parallelogram has vertices $A(2, 1, -4)$, $B(1, 0, -1)$ and $C(1, 2, 3)$. Determine possible coordinates of its fourth vertex D. Write the equation of the plane in which the parallelogram lies. Compute the area of the parallelogram.

3. Determine if the points $(0, -1, 0)$, $(1, 1, 1)$ and $\left(\frac{1}{2}, -1, \frac{1}{2}\right)$ are collinear.

4. Determine the angle between an edge of a cube and the longest adjacent diagonal.

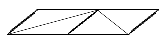
5. Calculate the area of a triangle with vertices $(0, 0, 0), (1, 0, 0), (2, 3, 4)$.

6. Write the equation of the plane which includes the point $(1, 7, 3)$ and is perpendicular to the line $L : x = 1 - 2t$, $y = 3 + t$, $z = 7$.

7. Find the equation of the plane containing the straight lines: $\dfrac{x-1}{3} = \dfrac{y+2}{-1} = \dfrac{z}{2}$ and $\dfrac{x-1}{2} = \dfrac{y+2}{3} = -z$.

8. Write the equation of the line which includes the point $(1, 7, 3)$ and is perpendicular to the the plane $x + 2y = 3$.

9. Write the equation of the line of intersection of the planes $x + 2y - z = 2$ and $3x + 2y + 2z = 7$.

10. (a) Prove that $|\mathbf{u} + \mathbf{v}|^2 + |\mathbf{u} - \mathbf{v}|^2 = 2|\mathbf{u}|^2 + 2|\mathbf{v}|^2$. (b) Prove that $(\mathbf{u} + \mathbf{v}) \times (\mathbf{u} - \mathbf{v}) = -2\,\mathbf{u} \times \mathbf{v}$. Explain the connection between equation (b) and this picture: 

11. Write the equations of the curves of intersection of the sphere $x^2 + y^2 + z^2 = 25$ and the cylinder $x^2 + y^2 = 16$.

12. Sketch these graphs. (State the name of each surface and include all intercepts.)
    (a) $x^2 + z^2 = 4$ (b) $z = \dfrac{y^2}{4}$ (c) $x^2 + \dfrac{y^2}{4} + z^2 = 1$ (d) $z = 4 - x^2 - y^2$
    (e) $x^2 + z^2 = y$ (f) $x^2 + z^2 = y^2$ (g) $x^2 - z^2 = y$ (h) $\dfrac{y^2}{4} + \dfrac{x^2}{4} - z^2 = 1$
    (i) $\dfrac{y^2}{4} - \dfrac{x^2}{4} - z^2 = 1$ (j) $x^2 - 16y^2 + 9z^2 = 0$ (k) $x^2 - 16y + 9z^2 = 0$
    (l) $x^2 + y^2 = z^2 + 9$ (m) $z^2 = 9 - 4x^2 - y^2$

13. State the name and draw rough sketches of each of the following surfaces:
    (a) $z = r$ (b) $9x^2 + y^2 + z^2 = 36$
    (c) $9x^2 - 16y^2 + z^2 = 144$ (d) $\rho = 3\cos\phi$
    (e) $z = 3 - r^2$ (f) $9x^2 - 16y^2 - z^2 = 144$

14. Sketch the region given in cylindrical coordinates:
    (a) $z^2 + r^2 = 9$ (b) $z + r^2 = 9$ (c) $r = 3$
    (d) $r = 4\cos\theta$ (e) $z = 2r$ (f) $r = \dfrac{6}{2 - \cos\theta}$

15. Sketch the region given in spherical coordinates:
    (a) $\rho = 5$ (b) $\theta = \dfrac{\pi}{6}$ (c) $\phi = \dfrac{\pi}{4}$ (d) $\rho = 5\cos\phi$ (e) $\rho\sin\phi = 8$

16. Write an equation in cylindrical coordinates for the ellipsoid $4x^2 + 4y^2 + z^2 = 1$.

17. What is the equation in Cartesian coordinates for the surface whose equation in spherical coordinates is $\rho = \sin\theta\sin\phi$?

18. Write the equation in spherical coordinates for the surface whose equation in Cartesian coordinates is $x^2 + y^2 + (z-1)^2 = 1$.

19. (a) Write the equation $x^2 - y^2 + z^2 = 1$ in cylindrical coordinates and simplify.
    (b) Write the equation $x^2 + y^2 = 9$ in spherical coordinates and simplify.

20. The angular momentum of a particle is $\mathbf{L}(t) = m\mathbf{r}(t) \times \mathbf{v}(t)$. Its torque is $\boldsymbol{\tau}(t) = m\mathbf{r}(t) \times \mathbf{a}(t)$. Prove $\frac{d}{dt}\mathbf{L}(t) = \boldsymbol{\tau}(t)$. Deduce that if $\boldsymbol{\tau}(t) = \mathbf{0}$ for all t, then $\mathbf{L}(t)$ is constant. (This is the law of conservation of angular momentum.)

21. A position vector of a particle (moving around an ellipse) is $\mathbf{r}(t) = \langle 3\cos t, 2\sin t\rangle$. Find the maximum and minimum values of the magnitude of its acceleration.

22. Prove that if a particle's speed is constant, then its acceleration is directed along $\mathbf{N}$.

23. Write the equation of the osculating circle of $y = \sin x$ at $\left(\frac{\pi}{2}, 1\right)$.

24. $\mathbf{r}(t) = \langle \cos t, \sin t, t\rangle$. Find a parametrization by arc length.

25. Determine $\mathbf{T}, \mathbf{N}$ and κ for the circular helix $\mathbf{r}(t) = a\cos t\,\mathbf{i} + a\sin t\,\mathbf{j} + bt\,\mathbf{k}$.

26. Find the tangential and normal components of acceleration for $\mathbf{r}(t) = \langle t - \sin t, 1 - \cos t\rangle$.

27. Write parametric equations for the tangent line at $\left(1, 1, \frac{2}{3}\right)$ to the curve $\mathbf{r}(t) = \langle t, t^2, \frac{2}{3}t^3\rangle$.

28. Determine the tangential and normal components of acceleration of a particle with position vector $\mathbf{r}(t) = \langle t, t, t^2\rangle$.

29. $\mathbf{r}(t) = \langle t\sin t, t\cos t, t^2 - 1\rangle$, $t \in [0, 2\pi]$. Compute: the velocity, speed and acceleration at $t = \pi$; the arc length; the tangential and normal components of acceleration at $t = \pi$; κ at $t = \pi$. Sketch the curve.

30. A particle P moves along a curve $\mathbf{r}(t) = (t - \frac{t^3}{3})\,\mathbf{i} + t^2\,\mathbf{j} + (t + \frac{t^3}{3})\,\mathbf{k}$. Find: the velocity vector $\mathbf{v}(t)$ and the unit tangent vector $\mathbf{T}(t)$; the unit normal vector $\mathbf{N}(t)$ and the curvature $\kappa(t)$; an integral for the length of the curve cut off by the planes $z = 0$ and $z = 12$.

31. A particle P moves along a curve $\mathbf{r}(t) = \langle 2\,\mathrm{e}^t\sin t, \mathrm{e}^t, -2\,\mathrm{e}^t\cos t\rangle$. Find: the velocity vector $\mathbf{v}(t)$ and the unit tangent vector $\mathbf{T}(t)$; the unit normal vector $\mathbf{N}(t)$ and the curvature $\kappa(t)$; an integral for the length of the curve cut off by the planes $y = 1$ and $y = e^2$.

32. At what point does $y = \mathrm{e}^x$ have maximum curvature? What is the curvature and the radius of curvature at that point? Find the equation of the osculating circle at the point; draw a sketch of the graph of the function and the osculating circle at the point.

33. Find $\mathbf{r}(t)$ if $\mathbf{r}'(t) = \sin t\,\mathbf{i} - \cos t\,\mathbf{j} + 2t\,\mathbf{k}$ and $\mathbf{r}(0) = \mathbf{i} + \mathbf{j} + 2\mathbf{k}$.