Big and Little

When considering the terms of infinite series (or even sequences), you will often find it useful in getting an intuitive grasp of what's going on if you remember what sorts of terms are bigger (or smaller) than what other sorts of terms. Here is a general summary of many standard sorts of expressions, which should help you. In the following table, terms of any one row are eventually "infinitely" smaller than terms from rows further down the table, in the sense that the limit of the quotient (higher row / lower row) will be 0 as $n \to \infty$. For simplicity, all terms are supposed to be positive (that could be weakened with appropriate use of absolute value signs), and generally n will refer to an integer, a to a real number > 1. In fact, the only place where n has to be an integer is in the expression $n!$ — in all other cases, n could be a real variable.

small
$(\ln(n))^k$
n^k
a^n
$n!$
n^n
LARGE

So, for example, $(\ln(n))^6 < \sqrt{n}$ and $5^{2n+1} < n!$ (for all sufficiently large n). More to the point,

$$\lim_{n\to\infty} \frac{(\ln(n))^6}{\sqrt{n}} = 0 \quad \text{and} \quad \lim_{n\to\infty} \frac{5^{2n+1}}{n!} = 0$$

Within each group (as suggested by the rows), relative size is pretty obvious, and is determined by the parameter k or a. So n^3 dominates (is bigger than) $\sqrt[3]{n}$, and 5^n dominates e^n.

In the following Appendix, you can find some exercises that will give you some first hand experience with the implicit limits in the claims above (including the two illustrated above), as well as sketches of proofs of the claims. We shall use a simple result (a consequence of the ratio test and the nth term test (a.k.a. the divergence test) for series), which you may not have seen before:

Theorem (The Ratio Test for Sequences). *For any sequence $\{a_n\}$, suppose the limit*

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L$$

exists (or is ∞). If $L < 1$, then $\lim a_n = 0$; if $L > 1$, then a_n diverges (to ∞ if the a_n are eventually positive). If $L = 1$ the test is inconclusive.

(A sketch of a proof of this fact will appear in the Appendix.)

But before that, here are two simple examples indicating how one would use the intuitions embodied in the table above.

Example 1. Does the series $\sum \frac{\ln n}{\sqrt{n^3}}$ converge or diverge?

Answer: Use the comparison test: since $\ln n < \sqrt[4]{n}$ (eventually), then

$$0 < \frac{\ln n}{\sqrt{n^3}} < \frac{\sqrt[4]{n}}{\sqrt{n^3}} = \frac{1}{n^{5/4}}$$

and since $\sum \frac{1}{n^{5/4}}$ is a converging p-series, then the original series converges too.

Without some understanding of the relative sizes of the terms in this series, one might think one had to use the integral test for this example. If you need some convincing, try the problem with the integral test: it works, but takes rather longer!

Example 2. Does $\sum \frac{2^n + n^2}{5^n + n^5}$ converge or diverge?

Answer: Using the "throw away the smaller terms" method, you can simply use the limit comparison test with $\sum \frac{2^n}{5^n} = \sum (\frac{2}{5})^n$, which converges (it's a geometric series with $r = \frac{2}{5}$); so the original series converges.

Appendix

Exercises:

1. Calculate $\lim_{n\to\infty} \frac{(\ln(n))^6}{\sqrt{n}}$.

2. Show that for any positive k, ℓ, $\lim_{n\to\infty} \frac{(\ln(n))^k}{n^\ell} = 0$

3. Calculate $\lim_{n\to\infty} \frac{\sqrt[3]{n}}{5^n}$

4. Show that for any positive k, any $a > 1$, $\lim_{n\to\infty} \frac{n^k}{a^n} = 0$

5. Calculate $\lim_{n\to\infty} \frac{5^{2n+1}}{n!}$.

6. Show that for any positive a, $\lim_{n\to\infty} \frac{a^n}{n!} = 0$

7. Calculate $\lim \frac{n!}{n^n}$.

8. Prove the Ratio Test for Sequences.

Answers:

1. Use L'Hôpital:

$$\begin{aligned}
\lim \frac{(\ln(x))^6}{\sqrt{x}} &= \lim \frac{6(\ln(x))^5}{x\,\frac{1}{2}\,x^{-1/2}} \\
&= \lim \frac{12(\ln(x))^5}{x^{1/2}} \\
&= \lim \frac{60(\ln(x))^4}{\frac{1}{2}x^{-1/2}} \\
&= \ldots \\
&= \lim \frac{46080}{x^{1/2}} \\
&= 0
\end{aligned}$$

2. Use L'Hôpital, as above. $\lim \dfrac{(\ln(x))^k}{x^\ell} = \dfrac{k}{\ell} \lim \dfrac{(\ln(x))^{k-1}}{x^\ell} = \ldots = 0$, since it's clear this will reduce the expression on top eventually to a constant, without altering the bottom, at which point the limit clearly $= 0$. (Formally, this would be proven by mathematical induction on k.)

3. Use the Ratio Test for Sequences:

$$\begin{aligned}
\lim_{n\to\infty} \left| \frac{\sqrt[3]{n+1}}{5^{n+1}} \frac{5^n}{\sqrt[3]{n}} \right| &= \frac{1}{5} \lim \sqrt[3]{\frac{n}{n+1}} \\
&= \frac{1}{5} \; < \; 1
\end{aligned}$$

and so $\lim \dfrac{\sqrt[3]{n}}{5^n} = 0$.

4. Use the Ratio Test for Sequences as above, to get $\lim \left| \dfrac{(n+1)^k}{a^{n+1}} \dfrac{a^n}{n^k} \right| = \dfrac{1}{a} < 1$ and so $\lim \dfrac{n^k}{a^n} = 0$.

5. Use the Ratio Test for Sequences:

$$\begin{aligned}
\lim \left| \frac{5^{2n+3}}{(n+1)!} \frac{n!}{5^{2n+1}} \right| &= \lim \frac{25}{n+1} \\
&= 0 \; < \; 1
\end{aligned}$$

so $\lim \dfrac{5^{2n+1}}{n!} = 0$.

6. Use the Ratio Test for Sequences as above, to get $\lim \left| \dfrac{a^{n+1}}{(n+1)!} \dfrac{n!}{a^n} \right| = \lim \dfrac{a}{n+1} = 0$, and so $\lim \dfrac{a^n}{n!} = 0$.

7. RT again (with a bit of L'Hôpital):
$\lim \dfrac{(n+1)!}{(n+1)^{n+1}} \dfrac{n!}{n^n} = \lim \left(\dfrac{n}{n+1} \right)^n = \mathrm{e}^{-1} < 1$ so $\lim \dfrac{n!}{n^n} = 0$.

8. If you know the Ratio Test for Series and the nth term test (also called the divergence test), then the Ratio Test for Sequences is simply a consequence of the fact that a convergent series must have its individual terms $\to 0$. But a more direct proof follows from the following consideration. If $L < 1$, then one can find an r, $L < r < 1$. Then eventually $\left|\frac{a_{n+1}}{a_n}\right| < r$, and so $|a_n|$ is bounded above by a geometric sequence with $0 < r < 1$ (and so which $\to 0$), and hence $a_n \to 0$ itself. (The case when $L > 1$ may be handled by considering the sequence $1/a_n$.)