Population growth: solutions without solving!

We have seen in class how to solve some simple differential equations, to get explicit formulas for the function y involved. But in some cases, this is perhaps difficult, but we would still like to derive some information about the type of solution from the equation itself, directly, without actually solving it. Here a few examples where this is not difficult, examples involving population growth, like the logistic example we saw in class (and in the textbook).

We'll start with a simple variant of the logistic equation, namely this:

$$\frac{dP}{dt} = k\left(1 - \frac{P}{M}\right)$$

where k and M are constants (and as before $P = P(t)$ is a function of t). This is not hard to solve, and I shall leave that to you as an exercise, and I suggest you graph some solutions (for some particular values of k and M, such as $k = 10$ and $M = 100$, and for some particular initial condition, such as $P(0) = 1$ or $P(0) = 150$, for example). What I want to point out here is a little different: just looking at the equation we can conclude some facts about its behaviour. For example, when do we see P tending to, or actually having, a horizontal tangent? In other words, when is $P' = 0$? That obviously means we must solve $k(1-P/M) = 0$, which is easy: $P = M$. It is equally clear that P' is positive (and so P is increasing) when $k(1 - P/M)$ is positive, and assuming k, M are positive, this is when $P < M$. Similarly, P is decreasing if $P > M$. So we have a function that either increases (as $t \to \infty$) to an "equilibrium value" $P = M$, or decreases to the equilibrium value of M, (depending on whether its starting value is less than or greater than M). Graphs of possible solutions are shown in Figure 1, using the values suggested above.

Let's look at the two examples done in the book from this perspective. First, the exponential growth example:

$$P' = kP$$

Here, since P is positive (and we shall assume k is too), P' is positive, so the function always increases, just as we saw when we solved this equation, regardless of the initial condition.

Next, the logistic equation

$$P' = kP\left(1 - \frac{P}{M}\right)$$

Again, it's easy to calculate the solutions to $P' = 0$, to get $P = 0$ or $P = M$. The first would apply if the population started out as 0, and not surprisingly, it would remain at 0 in that case; the second is evidently the equilibrium solution.

This is discussed in the textbook, in sections 9.1, 9.2, and a typical example is shown in Figure 2 on the next page.

Some exercises:

So: with this in mind, here are a few to try yourself. Find the rough behaviour (increasing, or decreasing?; equilibrium solution or solutions?) for the following models of population growth. I am not asking you to actually solve these differential equations (that would be rather more challenging!). Assume k, M, h are positive constants. If you wish, choose some values for these constants (and some initial conditions). Sketches of typical values are given in Figures 3–6.

1. $P' = kP\left(1 - \frac{P}{M}\right) - hP$
2. $P' = kP\left(1 - \frac{P}{M}\right) - hP^2$
3. $P' = kP\left(1 - \frac{P}{M}\right) - h$

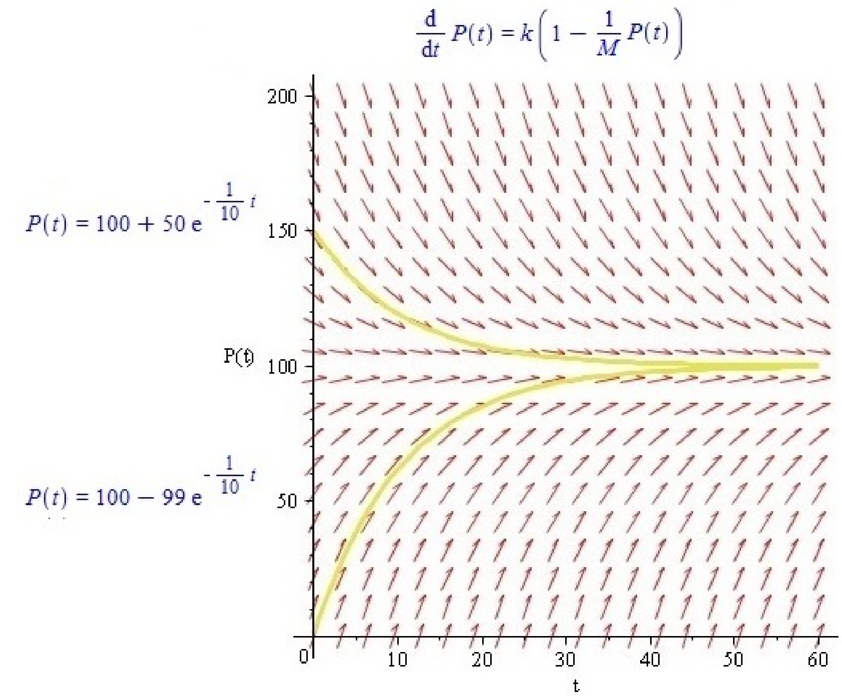

Figure 1: $\frac{dP}{dt} = k\left(1 - \frac{P}{M}\right)$ with $k = 10, M = 100$; two solutions highlighted: $P(0) = 1$ and $P(0) = 150$.

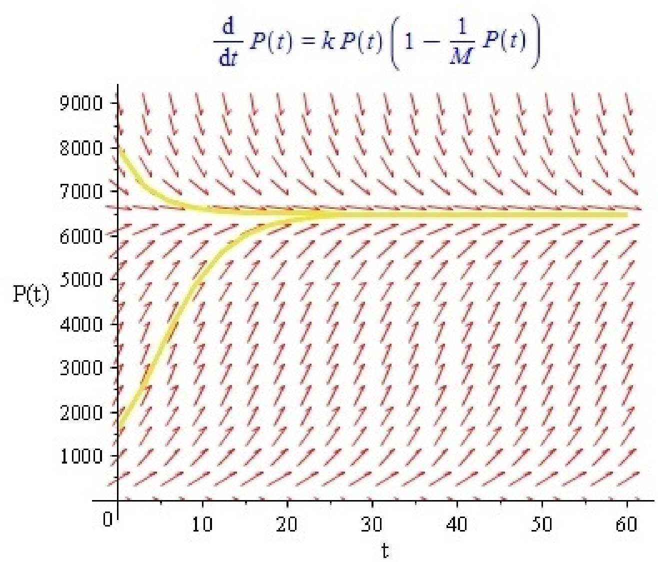

Figure 2: $\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right)$ with $k = 0.25, M = 6500$; two solutions highlighted: $P(0) = 1600$ and $P(0) = 8000$.

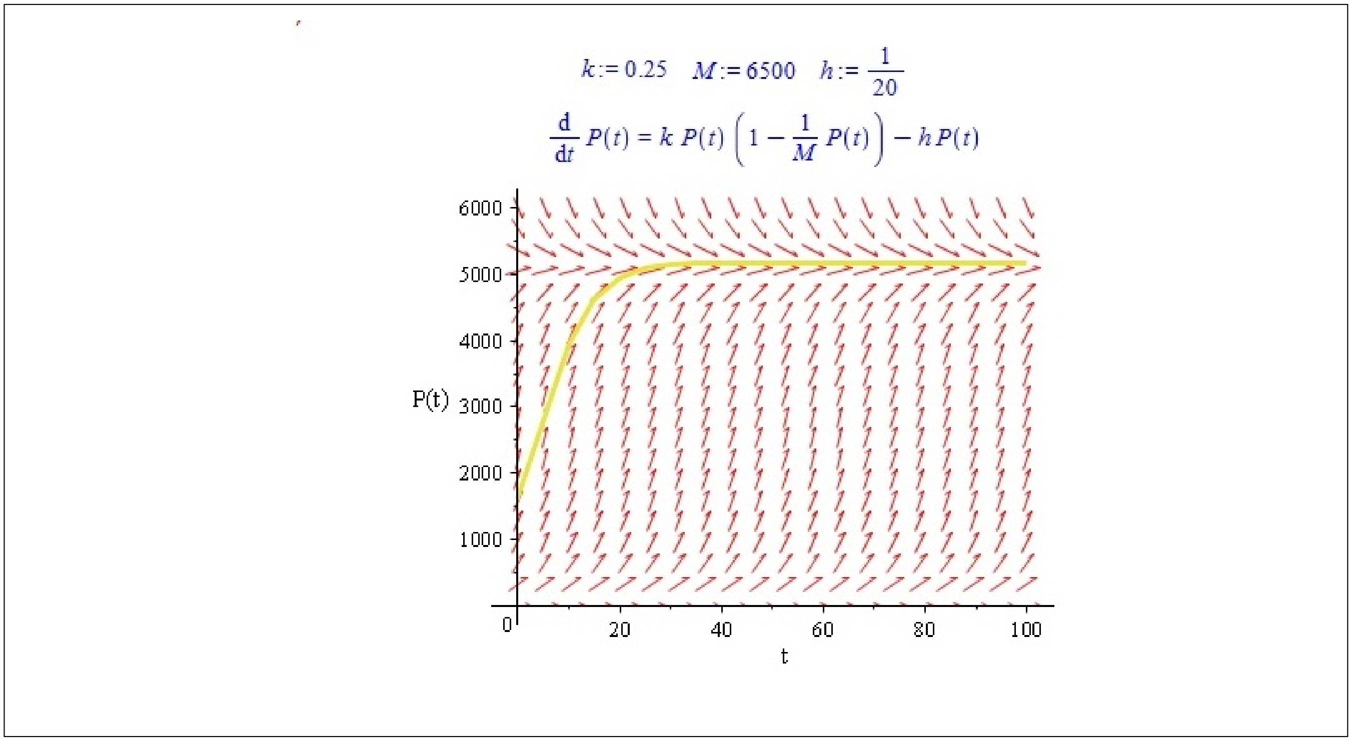

Figure 3: $\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right) - hP$

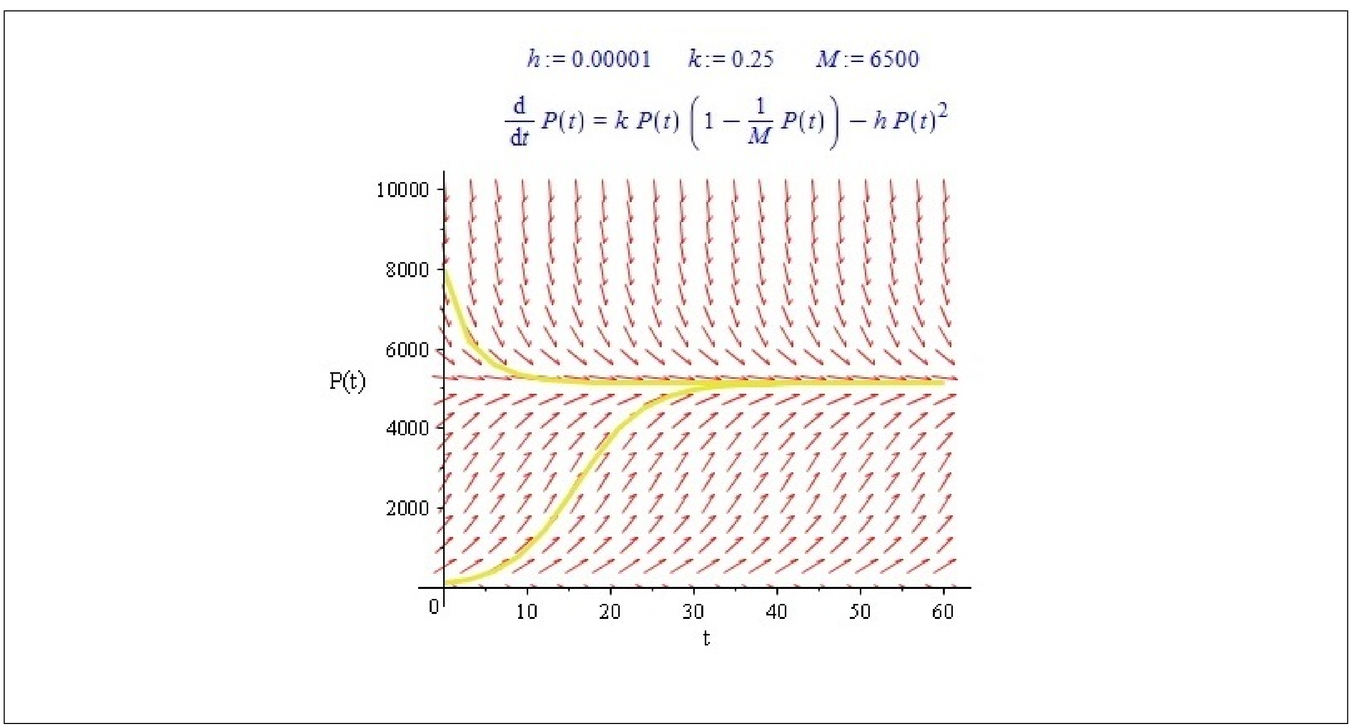

Figure 4: $\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right) - hP^2$

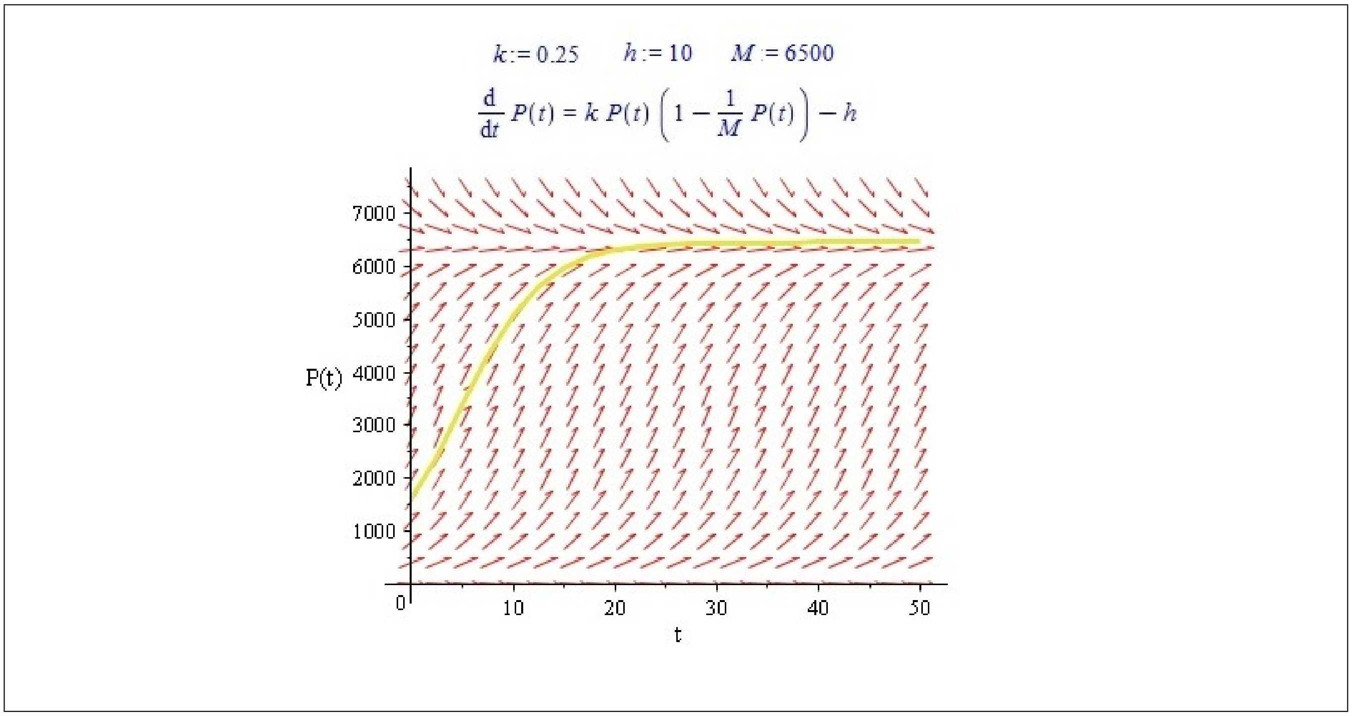

Figure 5: $\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right) - h$

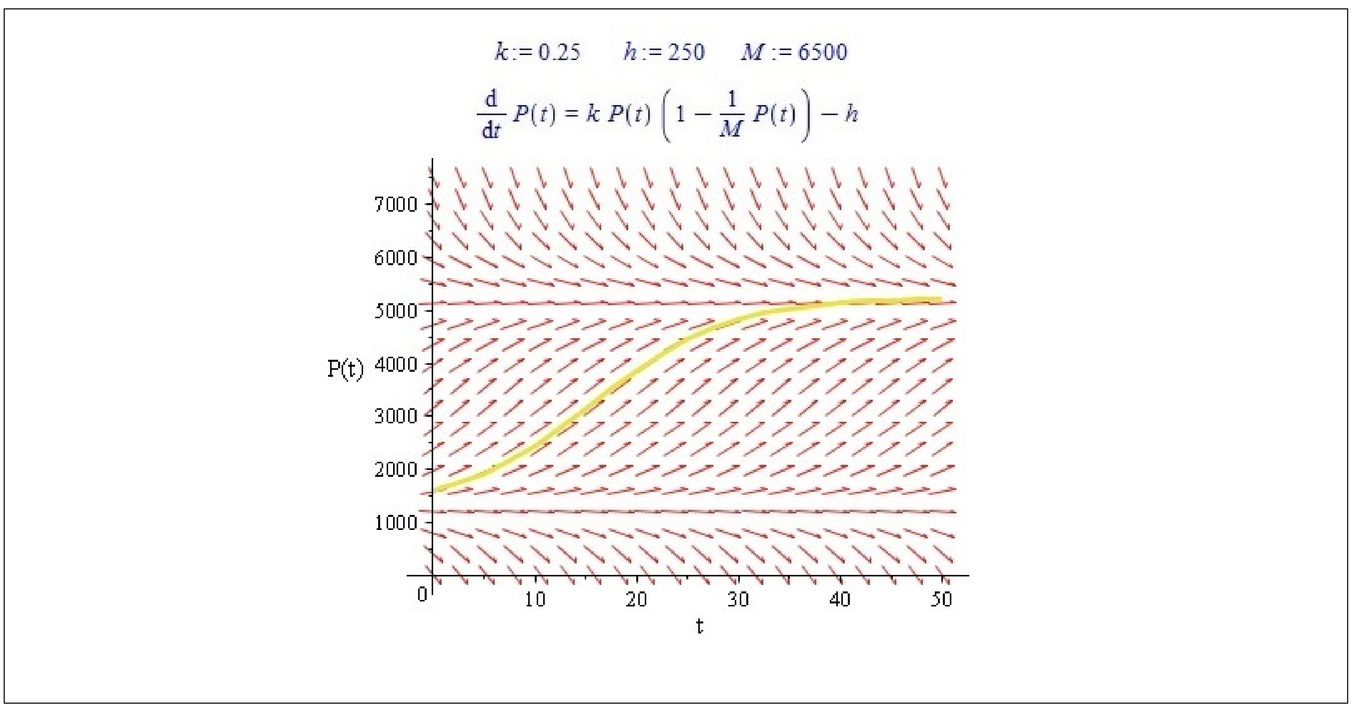

Figure 6: $\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right) - h$, different h, showing two equilibrium solutions