Calculus **II** (Maths 201–NYB) Quiz 4p **Answers**

1(a) $\int x\tan^2 x\,dx = \int x(\sec^2 x - 1)\,dx = \int x\sec^2 x\,dx - \int x\,dx$. Let $u = x$, $dv = \sec^2 x\,dx \Rightarrow du = dx$, $v = \tan x$.

Then by Equation 2, $\int x\sec^2 x\,dx = x\tan x - \int \tan x\,dx = x\tan x - \ln|\sec x|$, and thus,

$\int x\tan^2 x\,dx = x\tan x - \ln|\sec x| - \frac{1}{2}x^2 + C.$

1(b) $I = \int x\sin^3 x\,dx$. First, evaluate

$\int \sin^3 x\,dx = \int (1-\cos^2 x)\sin x\,dx \overset{c}{=} \int (1-u^2)(-du) = \int (u^2-1)\,du = \frac{1}{3}u^3 - u + C_1 = \frac{1}{3}\cos^3 x - \cos x + C_1.$

Now for I, let $u = x$, $dv = \sin^3 x \Rightarrow du = dx$, $v = \frac{1}{3}\cos^3 x - \cos x$, so

$$\begin{aligned} I &= \tfrac{1}{3}x\cos^3 x - x\cos x - \int \left(\tfrac{1}{3}\cos^3 x - \cos x\right)dx = \tfrac{1}{3}x\cos^3 x - x\cos x - \tfrac{1}{3}\int \cos^3 x\,dx + \sin x \\ &= \tfrac{1}{3}x\cos^3 x - x\cos x - \tfrac{1}{3}(\sin x - \tfrac{1}{3}\sin^3 x) + \sin x + C \\ &= \tfrac{1}{3}x\cos^3 x - x\cos x + \tfrac{2}{3}\sin x + \tfrac{1}{9}\sin^3 x + C \end{aligned}$$

1(c) Let $u = \sin 8x$, $dv = \cos 5x\,dx$, so $du = 8\cos 8x\,dx$, $v = \frac{1}{5}\sin 5x$. Then let $u = 8\cos 8x$, $dv = \frac{1}{5}\sin 5x\,dx$, so $du = -64\sin 8x\,dx$, $v = -\frac{1}{25}\cos 5x$. So

$$\int \sin 8x\cos 5x\,dx = \frac{1}{5}\sin 8x\sin 5x + \frac{8}{25}\cos 8x\cos 5x + \frac{64}{25}\int \sin 8x\cos 5x\,dx$$

Solving for $\int \sin 8x\cos 5x\,dx$, we get $\int \sin 8x\cos 5x\,dx = -\frac{5}{39}\sin 8x\sin 5x - \frac{8}{39}\cos 8x\cos 5x + C$

1(d) $\int \tan x\sec^3 x\,dx = \int \tan x\sec x\sec^2 x\,dx = \int u^2\,du$ $[u = \sec x,\ du = \sec x\tan x\,dx]$

$= \frac{1}{3}u^3 + C = \frac{1}{3}\sec^3 x + C$

1(e) Let $u = \ln\sqrt[3]{x}$, $dv = dx \Rightarrow du = \frac{1}{\sqrt[3]{x}}\left(\frac{1}{3}x^{-2/3}\right)dx = \frac{1}{3x}dx$, $v = x$. Then

$$\int \ln\sqrt[3]{x}\,dx = x\ln\sqrt[3]{x} - \int x\cdot\frac{1}{3x}dx = x\ln\sqrt[3]{x} - \frac{1}{3}x + C.$$

Second solution: Rewrite $\int \ln\sqrt[3]{x}\,dx = \frac{1}{3}\int \ln x\,dx$

Third solution: Substitute $y = \sqrt[3]{x}$, to obtain $\int \ln\sqrt[3]{x}\,dx = 3\int y^2\ln y\,dy$

1(f) Let $u = \frac{1-x}{2+x}$; then $du = \frac{-3}{(2+x)^2}$:

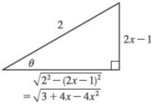
$$\int \frac{1}{(2+x)^2}\sqrt{\frac{1-x}{2+x}}\,dx = -\frac{1}{3}\int \sqrt{u}\,du = -\frac{2}{9}u^{3/2} = -\frac{2}{9}\left(\frac{1-x}{2+x}\right)^{3/2} + C$$

1g. Let $u = 1+\sqrt{x}$, so $\sqrt{x} = u-1$ and $du = \frac{1}{2\sqrt{x}}\,dx$, so $dx = 2(u-1)\,du$.

Then $\int \sqrt{1+\sqrt{x}}\,dx = 2\int (u-1)\sqrt{u}\,du = 2\int (u^{3/2} - u^{1/2})\,du = \frac{4}{5}(1+\sqrt{x}\,)^{5/2} - \frac{4}{3}(1+\sqrt{x}\,)^{3/2} + C.$

(Trig sub also works.)

1h. $3+4x-4x^2 = -(4x^2-4x+1)+4 = 2^2-(2x-1)^2.$



Let $2x-1 = 2\sin\theta$, so $2\,dx = 2\cos\theta\,d\theta$ and $\sqrt{3+4x-4x^2} = 2\cos\theta$.

Then

$$\int \frac{x^2}{(3+4x-4x^2)^{3/2}}\,dx = \int \frac{\left[\frac{1}{2}(1+2\sin\theta)\right]^2}{(2\cos\theta)^3}\cos\theta\,d\theta$$

$$= \frac{1}{32}\int \frac{1+4\sin\theta+4\sin^2\theta}{\cos^2\theta}\,d\theta = \frac{1}{32}\int (\sec^2\theta + 4\tan\theta\,\sec\theta + 4\tan^2\theta)\,d\theta$$

$$= \tfrac{1}{32}\int [\sec^2\theta + 4\tan\theta\,\sec\theta + 4(\sec^2\theta - 1)]\,d\theta$$

$$= \tfrac{1}{32}\int (5\sec^2\theta + 4\tan\theta\,\sec\theta - 4)\,d\theta = \tfrac{1}{32}(5\tan\theta + 4\sec\theta - 4\theta) + C$$

$$= \frac{1}{32}\left[5\cdot\frac{2x-1}{\sqrt{3+4x-4x^2}} + 4\cdot\frac{2}{\sqrt{3+4x-4x^2}} - 4\cdot\sin^{-1}\left(\frac{2x-1}{2}\right)\right] + C$$

$$= \frac{10x+3}{32\sqrt{3+4x-4x^2}} - \frac{1}{8}\sin^{-1}\left(\frac{2x-1}{2}\right) + C$$

1i. Let $u = 3+4x-4x^2$, so $du = -4(2x-1)\,dx$, so $(2x-1)\,dx = -\frac{1}{4}\,du$.

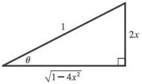
Then $\int \frac{2x-1}{(3+4x-4x^2)^{3/2}}\,dx = -\frac{1}{4}\int u^{-3/2}\,du = \frac{1}{2}(3+4x-4x^2)^{-1/2} + C.$

(1j) $\frac{x^2+x+1}{(x-1)^2(x^2+1)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{Cx+D}{x^2+1} \Rightarrow$

$x^2+x+1 = A(x-1)(x^2+1) + B(x^2+1) + (Cx+D)(x-1)^2$. Setting $x=1$ gives $B = \frac{3}{2}$. Equating the coefficients of x^3 gives $A = -C$. Equating the constant terms gives $1 = -A+B+D$, so $-A+D = -\frac{1}{2}$. Equating the coefficients of x gives $D = -\frac{1}{2}$. Hence $A = C = 0$.

$$\int \frac{x^2-2x-1}{(x-1)^2(x^2+1)}\,dx = \int \left[\frac{3}{2}\frac{1}{(x-1)^2} - \frac{1}{2}\frac{1}{x^2+1}\right]dx = -\frac{3}{2}\frac{1}{(x-1)} - \frac{1}{2}\tan^{-1}x + C.$$

(1k) Let $2x = \sin\theta$, where $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$. Then $x = \frac{1}{2}\sin\theta$, $dx = \frac{1}{2}\cos\theta\,d\theta$, and $\sqrt{1-4x^2} = \sqrt{1-(2x)^2} = \cos\theta$.



$$\int \sqrt{1-4x^2}\,dx = \int \cos\theta\left(\tfrac{1}{2}\cos\theta\right)d\theta = \tfrac{1}{4}\int (1+\cos 2\theta)\,d\theta$$

$$= \tfrac{1}{4}\left(\theta + \tfrac{1}{2}\sin 2\theta\right) + C = \tfrac{1}{4}(\theta + \sin\theta\,\cos\theta) + C$$

$$= \tfrac{1}{4}\sin^{-1}(2x) + \tfrac{1}{2}x\sqrt{1-4x^2} + C$$

2(a) This limit has the form $\infty \cdot 0$. We'll change it to the form $\frac{0}{0}$.

$$\lim_{x\to 0} \cot 2x \sin 6x = \lim_{x\to 0} \frac{\sin 6x}{\tan 2x} \overset{\text{H}}{=} \lim_{x\to 0} \frac{6\cos 6x}{2\sec^2 2x} = \frac{6(1)}{2(1)^2} = 3$$

2(b) This limit has the form $\infty - \infty$.

$$\lim_{x\to\infty} [\ln(x^7-1) - \ln(x^5-1)] = \lim_{x\to\infty} \ln\frac{x^7-1}{x^5-1} = \ln \lim_{x\to\infty} \frac{x^7-1}{x^5-1} \overset{\text{H}}{=} \ln \lim_{x\to\infty} \frac{7x^6}{5x^4} = \infty$$

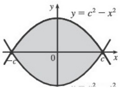
2(c) $\sqrt{e}$

3 We first assume that $c > 0$, since c can be replaced by $-c$ in both equations without changing the graphs, and if $c = 0$ the curves do not enclose a region. We see from the graph that the enclosed area A lies between $x = -c$ and $x = c$, and by symmetry, it is equal to four times the area in the first quadrant. The enclosed area is

$A = 4\int_0^c (c^2 - x^2)\,dx = 4\left[c^2x - \frac{1}{3}x^3\right]_0^c = 4\left(c^3 - \frac{1}{3}c^3\right) = 4\left(\frac{2}{3}c^3\right) = \frac{8}{3}c^3$

So $A = 576 \quad\Leftrightarrow\quad \frac{8}{3}c^3 = 576 \quad\Leftrightarrow\quad c^3 = 216 \quad\Leftrightarrow\quad c = \sqrt[3]{216} = 6$.

Note that $c = -6$ is another solution, since the graphs are the same.



4. $y = \tan\left(\frac{\pi}{x^2+4} - \frac{\pi}{12}\right)$

5. For $n = 1$, $a_1 = 0$ since $s_1 = 0$. For $n > 1$,

$$a_n = s_n - s_{n-1} = \frac{n-1}{n+1} - \frac{(n-1)-1}{(n-1)+1} = \frac{(n-1)n - (n+1)(n-2)}{(n+1)n} = \frac{2}{n(n+1)}$$

Also, $\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{1-1/n}{1+1/n} = 1$.

6. Use the Limit Comparison Test with $a_n = \frac{n+4^n}{n+6^n}$ and $b_n = \left(\frac{2}{3}\right)^n$

so the series $\sum_{n=1}^{\infty} \frac{n+4^n}{n+6^n}$ converges by comparison with $\sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n$, which is a convergent geometric series $\left[|r| = \frac{2}{3} < 1\right]$.

So the given series is absolutely convergent.

7. Telescoping: $s_n = \ln(n+2) - \ln(2) \to \infty$, so the series is not AC. But $\ln(\frac{x+2}{x+1})$ is decreasing (its derivative is $\frac{1}{x+2} - \frac{1}{x+1} < 0$). So by (AST) the series is conditionally convergent.

8. $T_4(x) = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \frac{5x^4}{128}$.
The Maclaurin series is $1 + \frac{1}{2}x + \sum_{n=2}^{\infty} \frac{1\cdot 3\cdot 5\cdots(2n+3)}{2^n\, n!} x^n$.