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Some differential equation problems

These questions are based on the problems in your textbook, as assigned in the course outline—you may find some hints (if necessary) in the book. Try to do these without looking at the hints and answers below.

1. A tank contains 1000 ℓ of pure water. A solution containing 0.2 kg of sugar per liter enters the tank at the rate 5 ℓ/min. The solution is mixed and drains from the tank at the same rate. Write a differential equation for this situation. How much sugar is in the tank initially? How much sugar is in the tank after t min? How much sugar is in the tank after 40 minutes? Find the concentration of the solution in the tank as time approaches infinity.

2. A tank contains 50 kg of salt dissolved in 1000 ℓ of water. Pure water enters the tank at the rate 10 ℓ/min. The solution is mixed and drains from the tank at the same rate. Write a differential equation for this situation. What is the concentration of the solution in the tank initially? Find the amount of salt in the tank after 1.5 hours.

3. Imagine a rumour spreads at a rate proportional to the product of the fraction F of the population who have heard the rumour and the fraction ($1-F$ obviously!) who have not heard the rumour. Write a differential equation for this situation. Suppose that a community has 500 inhabitants: at breakfast (8am) 20 people have heard the rumour, and 4 hours later, at noon exactly, half the community has heard the rumour. At what time will 80% of the community have heard it?

Answers:

1. $\frac{dA}{dt} = 1 - \frac{A}{200}$; $A(0) = 0; A(t) = 200(1 - \mathrm{e}^{-t/200})$; $A(40) = 200(1 - \mathrm{e}^{-1/5}) \approx 36.25$kg; $A \to 200$, so eventual concentration will be $\frac{1}{5}$kg/ℓ.

2. $\frac{dA}{dt} = -\frac{A}{100}$; initial concentration $= \frac{1}{20}$kg/ℓ; $A = 50\,\mathrm{e}^{-t/100}$, so $A(90) = 50\,\mathrm{e}^{-0.9} \approx 20.33$kg.

3. From $\frac{dF}{dt} = kF(1-F)$, we get $\ln|\frac{F}{1-F}| = kt + c$; so at $t = 0$, $c = \ln(1/24) = -\ln(24)$. At $t = 4$, we get $\ln(1) = 0 = 4k - \ln(24)$, so $k = \frac{1}{4}\ln(24)$. And hence $\ln\frac{F}{1-F} = (\frac{t}{4} - 1)\ln(24)$, so if $F = 0.8$, then $t = 4\ln(96)/\ln(24) \approx 5.74$. So 80% know the rumour at roughly 1:45pm.

Hints:

1. For this situation, let's call the amount of sugar $A(t)$ (in kg), t being time (in minutes). Then the rate of change in the amount of sugar is $\frac{dA}{dt} :=$ [rate in] − [rate out] and those rates may be calculated (as functions of t) from the data given.

2. In this case, note that the rate of salt in is 0, since the intake is pure water. Otherwise this problem is like the preceeding one.

3. Here the important thing to note is that the rate is proportional to the product $F(1-F)$, meaning that we need a constant of proportionality: $\frac{dF}{dt} = kF(1-F)$. So in this case our answer will involve two constants, which we will discover with the two bits of "intitial" data.