Notes for the WeBWorK differential equation word problems

Set 6 Problem 22

A tank contains 2580 L of pure water. A solution that contains 0.06 kg of sugar per liter enters tank at the rate 7 L/min The solution is mixed and drains from the tank at the same rate.

$y(0) = 0$ (kg)
Rate in: 0.06 kg/$\ell \times 7$ ℓ/min $= 0.42$ kg/min
Rate out: $y/2580$ kg/$\ell \times 7$ ℓ/min $= 7/2580$ kg/min
So $\frac{dy}{dt} = 0.42 - 7y/2580$, so $\ldots$ $-\frac{1}{7}\ln|1083.6 - 7y| = -t/2580 + C$.
At $t = 0$, $y = 0$, so $C = -\frac{1}{7}\ln(1083.6)$.
So $\frac{1}{7}\ln|(1083.6/(1083.6 - 7y)| = -t/2580, \ldots, y = 154.8(1 - \mathrm{e}^{-7t/2580})$

Set 6 Problem 23 (Below!)

The solution is given in the WeBWorK page (after the set closes!) - but it assumes you know the solution to the general logistic equation, so here are some direct details:
$\frac{dy}{dt} = ky(1-y); y(0) = 0.106, y(4) = 0.5$
Solving: $\frac{y}{1-y} = C\,\mathrm{e}^{kt}$, so, using $t = 0$, $C = .106/.894 = 53/447 (= 0.118568)$.
Also, $y = C(1-y)\,\mathrm{e}^{kt} = C\,\mathrm{e}^{kt} - y\,\mathrm{e}^{kt}$, and $\ldots$ so
$y = \frac{1}{1+\frac{1}{C}\,\mathrm{e}^{-kt}}$. Here $\frac{1}{C} = 447/53$, so: $y = \frac{1}{1+\frac{447}{53}\,\mathrm{e}^{-kt}} = \frac{1}{1+8.433962\,\mathrm{e}^{-kt}}$.

The rest is as set out in the WebWork solution (click "Solution"!).
(BTW - WeBWorK's decimal 8.43398 is actually not entirely correct—the correct decimal is given above (8.433962). Of course, this is not an issue if one uses the exact fraction 447/53.)

A rumor spreads through a school. Let $y(t)$ be the fraction of the population that has heard the rumor at time t and assume that the rate at which the rumor spreads is proportional to the product of the fraction y of the population that has heard the rumor and the fraction $1 - y$ that has not yet heard the rumor.

The school has 1000 students in total. At 8 a.m., 106 students have heard the rumor, and by noon, half the school has heard it. Using the logistic model explained above, determine how much time passes before 90% of the students will have heard the rumor.

Solution: Let $y(t)$ be the proportion of students that have heard the rumor at a time t hours after 8 a.m. In the logistic model, we have a capacity of $A = 1$ (100% of students) and an unknown growth factor of k. Hence,

$$y(t) = \frac{1}{1 - \frac{e^{-kt}}{C}}.$$

The initial condition $y(0) = 0.106$ allows us to determine the value of C:

$0.106 = \frac{1}{1-\frac{1}{C}}$; so $C = -0.118568$. Thus,

$$y(t) = \frac{1}{1 + 8.43398e^{-kt}}.$$

The condition $y(4) = 0.5$ now allows us to determine the value of k:

$\frac{1}{2} = \frac{1}{1+8.43398e^{-4k}}$; so $k \approx 0.533067\,\mathrm{hours}^{-1}$.

90% of the students have heard the rumor when $y(t) = 0.9$. Thus

$\frac{9}{10} = \frac{1}{1+8.43398e^{-0.533067t}}$

$t \approx 8.12186$ hours.

Thus, 90% of the students have heard the rumor after about 8.12186 hours.