Instructor: Dr. R.A.G. Seely

Test 1
(version for practice)

Cal I (S) (Maths 201–NYA)

We start with some review:

1. For each of the following functions, calculate the derivative $\frac{dy}{dx}$. Do not simplify your answers. (Use logarithmic differentiation where appropriate.)

   (a) $y = \sec(2x^7 - 7x^2)$
   (b) $y = \csc(x^5 + 2)\tan^2(x^3 - 8)$
   (c) $y = \mathrm{e}^{2x^3+1} - 3\ln(x^2+1)$
   (d) $y = \dfrac{\sqrt{8-3x^5}}{\ln(7x^4 - 5x)}$
   (e) $y = e^{x^3+1}\sin(x^2-1)$
   (f) $y = \ln^4\left(\cot\left(x^5 - 7x^4 + 17\right)\right)$
   (g) $y = (\ln x)^{x^2+1}$
   (h) $y = e^{x^3+1}\ln(3x^5 - \sec x)$
   (i) $y = \dfrac{x(4x^3-1)^{31}\sqrt[5]{x^7-2x^4}}{\sqrt[7]{3x-5}\,(x^6-5x^3)^{17}}$
   (j) $y = \cos(x^5+2)\tan(x^3-8)$
   (k) $y = 5^{2x} - 3\log_7(x^2+1)$
   (l) $y = \dfrac{\sqrt{8-3x^5}}{(7x^4-5x)^{32}}$
   (m) $y = (\sin x)^{x^2+1}$
   (n) $y = \log_7\left(x^2 + \ln x\right)$
   (o) $y = \dfrac{\ln(4x^3-1)}{\sqrt[7]{3x-5}}$
   (p) $y = \cot^4\left(\sqrt[3]{5x^3 - 2x^5}\right)$
   (q) $y = (x+1)^{\cos x}$
   (r) $y = \ln\left(\dfrac{2^x\cos^3(x^4-10)}{\sqrt[3]{3x^4+2x-5}}\right)$
   (s) $y = \left(3x^7 - 5x + 1\right)^{(x^3 - x)}$
   (t) $y = \mathrm{e}^{(1-x^2)} + \tan(2x+5)$
   (u) $y = \left(\dfrac{\left(2x^7 - 2\sec x + \mathrm{e}\right)^{24}\sin^2(x^4-2)}{(2x^5 - 3x^2 + 9)^6\sqrt[9]{\ln x - 5x^2 + 7}}\right)^{12}$
   (v) $y = \ln\left(\dfrac{\sqrt[3]{3x^4+2x-5}}{2^x\cos^3(x^4-10)}\right)$
   (w) $y = \log_5\left(x^2 + \ln x\right)$
   (x) $y = \mathrm{e}^{2\cos x} - 3\ln(x^4 + 10x - 2)$

2. And more:

   (a) $y = \csc^8\left(\sqrt{3x^4 - 25x^2}\right)$
   (b) $y = e^{x^2-1}\ln(x^3+1)$
   (c) $y = \mathrm{e}^{(x^2 + 3x^5)} + \tan(2x+5)$
   (d) $y = 7^{2x} - 3\log_5(x^4+1)$
   (e) $y = \tan^6\left(\ln(3x^8 - \mathrm{e}^{x^3-8})\right)$
   (f) $y = \log_5\left(\dfrac{\csc^3(3x^2-10x)}{5^x\sqrt{x^3-5}}\right)$
   (g) $y = e^{x^3+1}\ln(x^2-1)$
   (h) $y = \left(7x^3 - 5x + 1\right)^{(x^7 - x)}$
   (i) $y = \dfrac{\left(4x^9 - 9x^4 + 49\right)^8\sqrt[7]{6x^3-5x+7}}{\left(3x^5 - 2x^3 + \pi\right)^{24}\sin^4(x^2+1)}$

3. For each of the following, find the second derivative $\frac{d^2y}{dx^2}$. Simplify as appropriate.

   (a) $y = \ln(\sin x)$
   (b) $x\tan(x^2+1)$
   (c) $y = x\ln(x^2+1)$
   (d) $y = \csc(x^2)$
   (e) $y = \ln\left(x + \tan x\right)$
   (f) $y = \dfrac{x-4}{3x+5}$

4. For each of the following equations, find the first and second derivatives $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$:

   (a) $x^2y^5 - 3y^2 = \ln(xy^2) - 3$
   (b) $y^3 - x^5 = \ln(x^4y^5) + 23$
   (c) $\sin^3 y - x^5 = \ln(3x^4y^5) + \pi^2$

5. Find the slope and the equation of the tangent line to the following curves at the given point:

   (a) $y = \dfrac{2x-1}{x^2-1}$ at $x = 2$
   (b) $y = \dfrac{x^2-1}{2x-1}$ at $x = 2$
   (c) $xy^3 - x^2 = 3y + x^3$ at $(1,2)$
   (d) $y^3 - 2x^2y + 1 = \sin(x - y^2)$ at $(1,1)$
   (e) $y^2 - 2x^2 - 1 = \mathrm{e}^{2x-y}$ at $(1,2)$

And some "new" material:

6. Calculate the following limits (if they exist). If a limit does not exist, say so; if appropriate one-sided limits exist instead, state them explicitly; if a limit is infinite, state this explicitly.

(a) $\lim_{x\to+\infty} \dfrac{4-2x^5+3x^3}{5+7x^5-2x^3}$ (b) $\lim_{x\to-\infty} \dfrac{\sqrt{4x^6+4x^3-1}}{7x^3-x^2+9}$ (c) $\lim_{x\to1} \dfrac{\frac{3}{x+2}-1}{x-1}$

(d) $\lim_{x\to1} \dfrac{x^2+3x-4}{2x^2+x-3}$ (e) $\lim_{x\to-1} \dfrac{x^2-2x+3}{2x^2-9x-9}$ (f) $\lim_{x\to1} \dfrac{\sqrt[3]{1+x}-\sqrt[3]{1-x}}{x}$

(g) $\lim_{x\to0} \dfrac{2-\cos x}{2-x}$ (h) $\lim_{x\to3} \dfrac{x^2-2x-3}{2x^2-9x+9}$ (i) $\lim_{x\to1} \dfrac{x^2+3x-1}{2x^2+x-3}$

(j) $\lim_{x\to3^-} \dfrac{|5x-15|}{x-3}$ (k) $\lim_{x\to+\infty} \dfrac{4-x^2+3x^3}{7+x^2-5x^3}$ (l) $\lim_{x\to-\infty} \dfrac{\sqrt{2+9x^2}}{9+2x}$

(m) $\lim_{x\to1} \dfrac{\frac{2}{x+1}-1}{x-1}$ (n) $\lim_{x\to-1} \dfrac{x^2+3x-1}{2x^2+x-3}$ (o) $\lim_{x\to3} \dfrac{x^2-2x+3}{2x^2-9x+9}$

(p) $\lim_{x\to0} \dfrac{\sqrt{1+x}-\sqrt{1-x}}{x}$ (q) $\lim_{x\to1} \dfrac{1-\sqrt{2-x}}{x-1}$ (r) $\lim_{x\to-1} \dfrac{x^2+3x+2}{2x^2-x-3}$

(s) $\lim_{x\to0} \dfrac{x^2-x-2}{x^2-4x+4}$ (t) $\lim_{x\to0^-} \dfrac{\cos x}{2x}$ (u) $\lim_{x\to-\infty} \dfrac{4+5x}{\sqrt{5+4x^2}}$ (v) $\lim_{x\to0} \dfrac{1}{x\csc x}$

(w) $\lim_{x\to3} \dfrac{x^2+1}{\sqrt{3-x}}$ (x) $\lim_{x\to0} \dfrac{\sin 3x}{5x}$ (y) $\lim_{x\to3} \dfrac{\tan(x-3)}{x-3}$ (z) $\lim_{x\to\pi/2} \dfrac{\cos x}{x}$

7. For the function $f(x) = \begin{cases} 3kx+1 & \text{if } x<2 \\ x^2-k & \text{if } x\geq 2 \end{cases}$ find a value of k which makes the function continuous at $x=2$.

8. Let $f(x) = \begin{cases} ax^2+b & \text{if } x<1 \\ x-a & \text{if } x\geq 1 \end{cases}$. Find values of a and b that make $f(x)$ continuous at $x=1$.

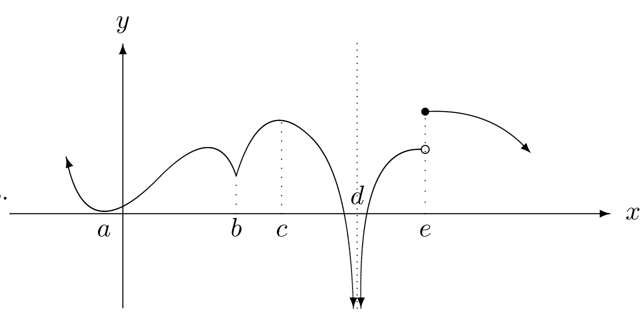
9. For the function $f(x) = \begin{cases} x^2+2 & \text{if } x<3 \\ 2x+k & \text{if } x\geq 3 \end{cases}$ find the value of k which makes the function continuous at $x=3$.

10. For the function $f(x) = \begin{cases} \sin(\frac{\pi}{2}x) & \text{if } x<3 \\ 5-2x & \text{if } 3\leq x\leq 5 \\ x^2 & \text{if } x>5 \end{cases}$ find all the values of x for which the function is discontinuous.

Specify if the discontinuity is removable or not. (Justify)

11. For the function $f(x) = \begin{cases} \dfrac{3x^2-5x-2}{3x^2-7x+2} & \text{if } x<2 \\ \dfrac{7}{5} & \text{if } x\geq 2 \end{cases}$

find all the values of x for which the function is discontinuous.



12. At the right is given the graph of a function; for each of the points $x = a, b, c, d, e$ state whether the function is (i) continuous and/or (ii) differentiable at the point.

13. The following functions may have one or more removable discontinuities: in each case, if so, define another function that removes them, but is otherwise the same as the given function.

(a) $f(x) = \dfrac{2x^3-x^2-x}{4x^3-x}$ (b) $f(x) = \dfrac{9x^3-x}{3x^3-2x^2-x}$

14. For each of the following, draw a sketch, if possible, of (the graph of) a function that satisfies the stated condition. If you think that the stated condition is impossible, say so and explain why (briefly!).

(a) The function must be continuous everywhere, but not differentiable at $x=0$.
(b) The function must be differentiable everywhere, but not continuous at $x=0$.
(c) The function must not be continuous at $x=0$, but $\lim_{x\to0} f(x)$ must be defined.

15. At a certain moment each edge of a cube is 10 cm long, and the surface area is increasing at a rate of 2.25 cm^2/sec. How fast is the volume of the cube increasing?

16. There is a picture of a square on a computer screen; the length of each side of the square is increasing at the rate of 0.25 cm/sec. Find the rate at which the area of the square is increasing when each side is 7.50 cm long.

17. An oil spill spreads in a circle whose radius increases at a constant rate of 10 m/sec. How rapidly is the area of the spill increasing when the radius is 120 m? (Hint: the area of a circle is $A = \pi r^2$.)

18. A conical water tank is being drained at a constant rate. The tank is 15 m high and 8m in diameter (at its top). The water level is falling at a rate of 75 cm/min when the level is 6 m. Find the rate at which the tank is being emptied. (Hint: $V = \frac{1}{3}\pi r^2 h$)

19. Another conical water tank is being filled at a rate of 20 ft^3/hr. This tank is 7 ft high with a 6 ft radius at the top. What is the rate at which the water level is rising when the water surface is 5 ft in diameter?

20. In an engine cylinder, the relationship between the pressure p (in kPa) and volume v (in cm^3) of the gas vapor is given by the equation $pv^2 = K$, for a constant K. At a certain time, the pressure and volume are determined to be $p = 4000$ kPa, $v = 80$ cm^3, and the volume is increasing at a rate of 800 cm^3/sec. At that time, what is the rate of change (with respect to time) of the pressure?

21. An circular oil spill is increasing in size, so that the area is is increasing at a rate of 50 cm^2/min. What is the rate of change (with respect to time) of the radius at the moment when the radius is 15 cm?

22. Let θ (in radians) be an acute angle in a right angled triangle, and let x, y be the lengths of the sides adjacent and opposite θ respectively. Suppose that x and y vary with time t. (a) What is $\frac{d\theta}{dt}$ in terms of $\frac{dx}{dt}$ and $\frac{dy}{dt}$? (Hint?: think of $\tan\theta$.) (b) At a certain instant, $x = 2$ cm and is increasing at 1 cm/s, while $y = 2$ cm and is decreasing at $\frac{1}{4}$ cm/s. How fast is θ changing at that instant.

23. A ladder 5 m long is leaning against a wall. The base of the ladder is sliding away from the wall at a rate of 1 m/s. How fast is the top of the ladder sliding down the wall at the instant when its base is 3 m away from the wall?

24. An inflated spherical balloon is punctured and gradually loses air at a constant rate of 10 cm^3/min. If the original diameter was 20 cm, find the rate at which the diameter is decreasing when it is: (a) 15 cm in diameter; (b) 10 cm in diameter; (c) half its original volume.

25. Find the values of x where the (absolute) minimum and maximum values of the function $f(x) = (x-2)^{\frac{2}{3}}$ occur on the interval $[1, 10]$.

26. Find the values of x where the (absolute) minimum and maximum values of the function $f(x) = x + \frac{1}{x}$ occur on the interval $[\frac{1}{4}, 8]$.

27. Find the values of x where the (global) minimum and maximum values of the function $f(x) = x\sqrt{1+x}$ occur on the interval $[-1, 1]$.

28. Find all (local) maximums and minimums for the function $f(x) = x^4 - 18x^2 + 9$. (Show clearly how you determine which is which (max or min?).)