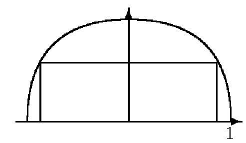
Cal I (S) (Maths 201–NYA)

The Dobson Files also give an excellect selection of problems, especially optimization and graphing problems (with answers).

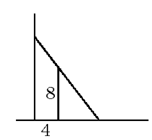
1. Sketch the graph of a function $f(x)$ having all of the following properties.
   (a) f has horizontal asymptotes at $y = -2$ and $y = 2$.
   (b) f has a vertical asymptote at $x = 1$.
   (c) f is continuous but not differentiable at $x = -2$.
   (d) f has a removable discontinuity at $x = 0$.
   (e) f is increasing on the intervals $-\infty < x < -2$, $-1 < x < 0$, $0 < x < 1$, $1 < x < 2$.
   (f) f is decreasing on the intervals $-2 < x < -1$, $2 < x < +\infty$.
   (g) f is concave up on the intervals $-\infty < x < 1$, $3 < x < +\infty$.
   (h) f is concave down on the interval $1 < x < 3$.

2. For the following functions, find all intercepts and asymptotes (if possible), determine at what points (x, y) the function has relative extrema and points of inflection. On what intervals is the function increasing? On what intervals is the function decreasing? Sketch a graph of the function; make sure your graph clearly illustrates all these features.
   (a) $f(x) = x + \dfrac{4}{x^2}$ (b) $f(x) = 5x^6 - 3x^5 + 10$ (c) $f(x) = x^4 - \frac{8}{3}x^3 + 5$
   (d) $f(x) = \dfrac{x}{(x+1)^2}$ (e) $f(x) = \dfrac{x}{x^2+4}$ (f) $f(x) = \dfrac{x^2-1}{(x-2)^2}$
   (g) $f(x) = x^{1/3}(x+1)$ (h) $f(x) = x\sqrt{1-x^2}$ (i) $f(x) = x\,\mathrm{e}^x$
   (j) $f(x) = x^4 - 5x^2 + 4$ (k) $f(x) = \dfrac{x-4}{x^2}$ (l) $f(x) = \sin x + \cos x$

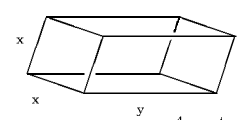
3. A rectangle is to be inscribed in a semi-circle of radius = 1 as shown in the figure; find the dimensions that give the rectangle with the largest possible area. (Hint: write the base as going from $-x$ to x, the height as y, and figure what is the equation of the circle.) ($\sqrt{2}$ by $1/\sqrt{2}$)



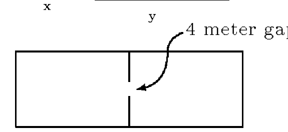
4. A fence 8 ft tall runs parallel to a tall building at a distance of 4 ft from the building. What is the length of the shortest ladder that will reach from the ground over the fence to the wall of the building? (16.65 ft)



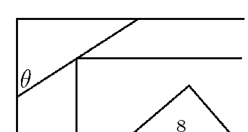
5. A box-shaped wire frame is made from a wire 1200 cm long (by cutting it into pieces to make the frame): the frame is supposed to consist of two identical squares, connected by four straight pieces of equal length (as shown at right). What should the dimensions of the box be in order to maximize the volume? (100 by 100)



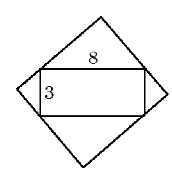
6. Joe has 200 meters of fencing with which to enclose two rectangular areas as shown in the figure. There is a 4 meter gap in the central divider to allow the installation of a special gate. What dimensions should be used so that the enclosed area is maximal? (34 by 51)



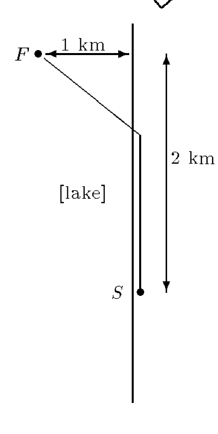
7. A steel pipe is being carried down a hallway 9 ft wide. At the end of the hall there is a right-angled turn into a narrower hallway 6 ft wide. What is the length of the longest pipe that can be carried horizontally around the corner? (Hint: What is the shortest line which touches the outside walls of the two halls and also touches the inside corner at the turn?) (21.07 ft)



8. Find the maximum area of a rectangle that can be circumscribed about a (fixed) rectangle of length 8 and width 3. (60.5)



9. A fisherman is in a rowboat on a lake, 1 km from shore. He has to reach a shop 2 km downstream, along the shore. He can row at 5 km/hr, and he can run at 13 km/hr. At what point on the shore should he land in order to get to the shop as quickly as possible. (5/12 km downstream)



10. A closed cylindrical can of volume 1 m^3 is to be designed to minimize cost. The cost of the top and bottom of the can is 80 cents per square meter, and the cost of the sides is 50 cents per square meter. Find the dimensions of the can with minimum cost. ($r = 0.46$, $h = 1.48$)

11. Evaluate the following:

(a) $\displaystyle\int \frac{1-3x^4+5\sqrt{x}}{x}\,dx$ (b) $\displaystyle\int x(x-1)^2\,dx$ (c) $\displaystyle\int \frac{1-\cos t}{\sin^2 t}\,dt$

(d) $\displaystyle\int (\sqrt[4]{x}-5^x)\,dx$ (e) $\displaystyle\int_1^3 \left(x^2-\frac{1}{x^2}\right)\,dx$ (f) $\displaystyle\int_1^e \left(1-\frac{1}{x}\right)\,dx$

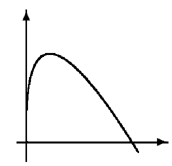
(g) $\displaystyle\int x(x+1)^2\,dx$ (h) $\displaystyle\int \frac{4x^3+6\sqrt{x}-1}{x}\,dx$ (i) $\displaystyle\int (\sqrt[5]{x}-\mathrm{e}^x)\,dx$

(j) $\displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{1-\sin\theta}{\cos^2\theta}\,d\theta$ (k) $\displaystyle\int \left(\frac{1}{\pi}+\frac{1}{x}\right)\,dx$ (l) $\displaystyle\int_1^3 \left(\frac{1}{x^2}-x^2\right)\,dx$

12. (a) Given $\int f(x)\,dx = 3x^2+e^{x^2+1}+c$, what is the function $f(x)$?

(b) Given $y''=4x^2-2$ together with the information $y'(1)=0,\ y(1)=1$, find the function y.

(c) Given $y''=\sin x+2$ together with the information $y'(\pi)=2\pi,\ y(\pi)=0$, find the function y.



13. (a) Find the area under the curve $y=\sqrt[3]{x}-x$, above the x-axis, between $x=0$ and $x=1$. (The graph is shown at right.)

(b) Find the area between the curves $x=1,\ x=9,\ y=x,$ and $y=\sqrt{x}$. Draw a rough sketch of the region concerned.

(c) The curve $y=x^4-5x^2+4$ intersects the x-axis to form three closed regions: find the total area of these regions.

14. Define a function $f(x)=\displaystyle\int_0^x \frac{t^2}{1+t^2}\,dt$. Calculate $f'(x)$ and $f''(x)$; sketch the graph of $f(x)$. (You may want to know that $f(1)=1-\frac{\pi}{4}$.)

15. Once, when the Tortoise and the Hare were having a race (along a straight road), the Tortoise passed the Hare twice. Prove that at some time during the race their accelerations were equal. State clearly what assumptions you are making, and what facts (*e.g.* from Calculus) you are using.