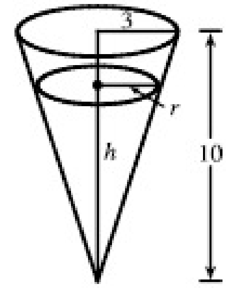
Workshop 1 Solutions

1 Given $dV/dt = 2$, find dh/dt when $h = 5$. $V = \frac{1}{3}\pi r^2 h$ and, from similar triangles, $\frac{r}{h} = \frac{3}{10} \quad \Rightarrow \quad V = \frac{\pi}{3}\left(\frac{3h}{10}\right)^2 h = \frac{3\pi}{100}h^3$, so

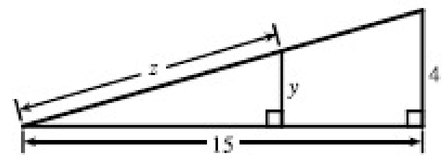
$$2 = \frac{dV}{dt} = \frac{9\pi}{100}h^2\,\frac{dh}{dt} \quad \Rightarrow \quad \frac{dh}{dt} = \frac{200}{9\pi h^2} = \frac{200}{9\pi\,(5)^2} = \frac{8}{9\pi} \text{ cm/s}$$

when $h = 5$.



2 We are given $dz/dt = 30$ ft/s. By similar triangles, $\frac{y}{z} = \frac{4}{\sqrt{241}} \quad \Rightarrow$

$$y = \frac{4}{\sqrt{241}}z, \text{ so } \frac{dy}{dt} = \frac{4}{\sqrt{241}}\,\frac{dz}{dt} = \frac{120}{\sqrt{241}} \approx 7.7 \text{ ft/s.}$$



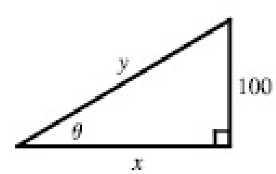
3 Let (b, c) be on the curve, that is, $b^{2/3} + c^{2/3} = a^{2/3}$. Now $x^{2/3} + y^{2/3} = a^{2/3} \quad \Rightarrow \quad \frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3}\,\frac{dy}{dx} = 0$, so $\frac{dy}{dx} = -\frac{y^{1/3}}{x^{1/3}} = -\left(\frac{y}{x}\right)^{1/3}$, so at (b, c) the slope of the tangent line is $-(c/b)^{1/3}$ and an equation of the tangent line is $y - c = -(c/b)^{1/3}(x - b)$ or $y = -(c/b)^{1/3}x + (c + b^{2/3}c^{1/3})$. Setting $y = 0$, we find that the x-intercept is $b^{1/3}c^{2/3} + b = b^{1/3}(c^{2/3} + b^{2/3}) = b^{1/3}a^{2/3}$ and setting $x = 0$ we find that the y-intercept is $c + b^{2/3}c^{1/3} = c^{1/3}(c^{2/3} + b^{2/3}) = c^{1/3}a^{2/3}$. So the length of the tangent line between these two points is

$$\sqrt{(b^{1/3}a^{2/3})^2 + (c^{1/3}a^{2/3})^2} = \sqrt{b^{2/3}a^{4/3} + c^{2/3}a^{4/3}} = \sqrt{(b^{2/3} + c^{2/3})a^{4/3}}$$
$$= \sqrt{a^{2/3}a^{4/3}} = \sqrt{a^2} = a = \text{constant}$$

4 We are given $dx/dt = 8$ ft/s. $\cot\theta = \frac{x}{100} \quad \Rightarrow \quad x = 100\cot\theta \quad \Rightarrow$

$$\frac{dx}{dt} = -100\csc^2\theta\,\frac{d\theta}{dt} \quad \Rightarrow \quad \frac{d\theta}{dt} = -\frac{\sin^2\theta}{100}\cdot 8. \text{ When } y = 200,\ \sin\theta = \frac{100}{200} = \frac{1}{2} \quad \Rightarrow$$

$$\frac{d\theta}{dt} = -\frac{(1/2)^2}{100}\cdot 8 = -\frac{1}{50} \text{ rad/s.}$$ The angle is decreasing at a rate of $\frac{1}{50}$ rad/s.



Workshop 1 Solutions

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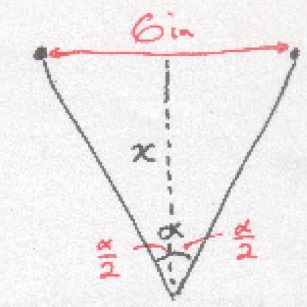
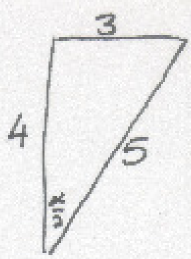




after 4 sec $x = 4$ (& by Pythagoras the length of ½ rubber band = 5)

So $\tan\frac{\alpha}{2} = \frac{3}{x}$

or $\cot\frac{\alpha}{2} = \frac{x}{3}$

So (take $\frac{d}{dt}$:) $-\frac{1}{2}\csc^2\frac{\alpha}{2}\frac{d\alpha}{dt} = \frac{1}{3}\frac{dx}{dt} = \frac{1}{3}$

So $\frac{d\alpha}{dt} = -\frac{2}{3}\sin^2\frac{\alpha}{2}$

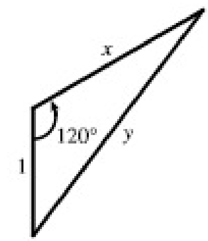
$= -\frac{2}{3}\left(\frac{3}{5}\right)^2 = \frac{-6}{25}$

6 We are given that $\frac{dx}{dt} = 300$ km/h. By the Law of Cosines,

$y^2 = x^2 + 1^2 - 2(1)(x)\cos 120^\circ = x^2 + 1 - 2x\left(-\frac{1}{2}\right) = x^2 + x + 1$, so

$2y\frac{dy}{dt} = 2x\frac{dx}{dt} + \frac{dx}{dt} \quad \Rightarrow \quad \frac{dy}{dt} = \frac{2x+1}{2y}\frac{dx}{dt}$. After 1 minute, $x = \frac{300}{60} = 5$ km $\quad \Rightarrow$

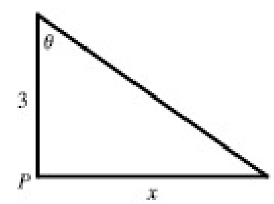
$y = \sqrt{5^2 + 5 + 1} = \sqrt{31}$ km $\quad \Rightarrow \quad \frac{dy}{dt} = \frac{2(5)+1}{2\sqrt{31}}(300) = \frac{1650}{\sqrt{31}} \approx 296$ km/h.



7 We are given that $\frac{d\theta}{dt} = 4(2\pi) = 8\pi$ rad/min. $x = 3\tan\theta \quad \Rightarrow$

$\frac{dx}{dt} = 3\sec^2\theta\,\frac{d\theta}{dt}$. When $x = 1$, $\tan\theta = \frac{1}{3}$, so $\sec^2\theta = 1 + \left(\frac{1}{3}\right)^2 = \frac{10}{9}$

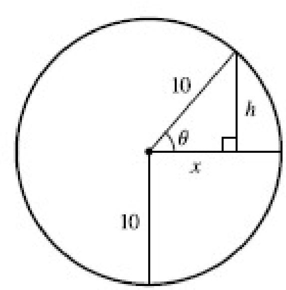
and $\frac{dx}{dt} = 3\left(\frac{10}{9}\right)(8\pi) = \frac{80}{3}\pi \approx 83.8$ km/min.



8 We are given that $\frac{d\theta}{dt} = \frac{2\pi \text{ rad}}{2 \text{ min}} = \pi$ rad/min. By the Pythagorean Theorem, when

$h = 6$, $x = 8$, so $\sin\theta = \frac{6}{10}$ and $\cos\theta = \frac{8}{10}$. From the figure, $\sin\theta = \frac{h}{10} \quad \Rightarrow$

$h = 10\sin\theta$, so $\frac{dh}{dt} = 10\cos\theta\,\frac{d\theta}{dt} = 10\left(\frac{8}{10}\right)\pi = 8\pi$ m/min.



Instructor: Dr. R.A.G. Seely | Workshop 1

(Oct 2016) | (version A)

Cal I (S) (Maths 201–NYA)

The quickies about derivatives:

1. True/False: (justify!) If $f(x)$ is differentiable:

   (a) $\dfrac{d}{dx}\left(\sqrt{f(x)}\right) = \dfrac{f'(x)}{2\sqrt{f(x)}}$ **True** (b) $\dfrac{d}{dx}\left(f(\sqrt{x})\right) = \dfrac{f'(x)}{2\sqrt{x}}$ **False**

2. (a) $y' = -\dfrac{1}{3}(x+\sqrt{x})^{-4/3}\left(1+\dfrac{1}{2\sqrt{x}}\right)$

   (b) $y' = \dfrac{(x^2+1)^4}{(2x+1)^3(3x-1)^5}\left(\dfrac{8x}{x^2+1} - \dfrac{6}{2x+1} - \dfrac{15}{3x-1}\right)$

   (c) $y' = -\sin(x)e^{\cos x} - \sin(e^x)e^x$

Postscript:
Alternate Solution to Q2 about the waterskier:

(I'll use the notation of Stewart's solution to Q100 on p.269 — as given on Lea!)

Although the method of similar triangles gives the answer "immediately", since $\frac{y}{z} = \frac{4}{\sqrt{241}}$ so that $\frac{dy}{dt} = \frac{4}{\sqrt{241}}\frac{dz}{dt} = \frac{120}{\sqrt{241}}$, one could also use Pythagoras as well.

$\frac{x}{y} = \frac{15}{4}$ so $x = \frac{15}{4}y$, so $z^2 = x^2 + y^2 = (\frac{4^2+15^2}{4^2})y^2$ so $2z\frac{dz}{dt} = 2y\frac{4^2+15^2}{4^2}\frac{dy}{dt}$, and hence $\frac{dy}{dt} = \frac{z}{y}\frac{4^2}{4^2+15^2}\frac{dz}{dt} = \frac{4\cdot 30}{\sqrt{4^2+15^2}} = \frac{120}{\sqrt{241}}$