Want to show: $\sqrt[3]{1+x} < 1 + \frac{1}{3}x$

Proof: Let $f(x) = \sqrt[3]{1+x}$, so $f'(x) = \frac{1}{3}(1+x)^{-2/3}$.
Note that f is continuous and differentiable on $[0, \infty)$, $(-1, \infty)$ in fact.
Note also that $f'(x) < \frac{1}{3}$ for any $x > 0$ (since $(1+x)^{-2/3} < 1$ for any $x > 0$).

For any $x \geq 0$, by the MVT there's a c (between 0 and x) so that

$$f'(c) = \frac{f(x) - f(0)}{x - 0}$$

so that

$$f(x) - 1 = f'(c) \cdot x$$

and so

$$f(x) = 1 + f'(c) \cdot x \; < \; 1 + \tfrac{1}{3}x$$

i.e.

$$\sqrt[3]{1+x} < 1 + \tfrac{1}{3}x$$

for any $x > 0$. (QED)

The answer as given in the exam:

13. Use the Mean Value Theorem to show that $\sqrt[3]{1+x} < 1 + \frac{1}{3}x$ for all $x > 0$.
(If it helps, you may use the fact that this is equivalent to proving that $\sqrt[3]{1+x} - (1 + \frac{1}{3}x) < 0$ for all $x > 0$.)

Answer

OPTION A: If $f(x) = (1+x)^{1/3}$ then f is differentiable (and thus continuous) on $(-1, \infty)$. For $x > 0$, the mean value theorem applied to f on $[0, x]$ yields a real number x_0 such that $0 < x_0 < x$ and $f(x) = f(0) + f'(x_0)(x - 0) = 1 + \frac{1}{3}x(1+x_0)^{-2/3}$ $< 1 + \frac{1}{3}x$, since $(1+x_0)^{-2/3} < 1$.

OR **OPTION B:** Let $f(x) = \sqrt[3]{1+x} - (1 + \frac{1}{3}x)$, so $f'(x) = \frac{1}{3\sqrt[3]{(1+x)^2}} - \frac{1}{3}$. Note that f is continuous on $[0, \infty)$ and differentiable on $(0, \infty)$. The MVT states that there must exist a point $x_0 \in (0, a)$ such that $f'(x_0) = \frac{f(a) - f(0)}{a - 0} = \frac{f(a)}{a}$ for any interval $(0, a)$, which implies that $a \cdot f'(x_0) = f(a)$ must hold for any $a > 0$. However, $f'(x)$ is always negative for $x > 0$, so $a \cdot f'(x_0)$ (and therefore $f(a)$) must always be less than zero when $a > 0$, i.e. $\sqrt[3]{1+x} < 1 + \frac{1}{3}x$ for all $x > 0$.