Integration by Parts: The "Table Method"

1 The Method

The table is made up of several "levels", beginning with the original choice of u and dv.

$$
\begin{array}{llll}
 & \text{Differentiate } u \text{ to get } du & & \text{Integrate } dv \text{ to get } v \\
(1)+ & \left[\begin{array}{rcl} u &=& \ldots \\ du &=& \ldots\ dx \end{array}\right. & & \begin{array}{rcl} dv &=& \ldots\ dx \\ v &=& \ldots \end{array} \\
 & \quad\Downarrow \ \text{(just rearrange)} & & \quad\Downarrow \ \text{(just rearrange)} \\
(2)- & \left[\begin{array}{rcl} u &=& \ldots \\ du &=& \ldots\ dx \end{array}\right. & & \begin{array}{rcl} dv &=& \ldots\ dx \\ v &=& \ldots \end{array} \\
 & \quad\Downarrow \ \text{(just rearrange)} & & \quad\Downarrow \ \text{(just rearrange)} \\
(3)+ & \left[\begin{array}{rcl} u &=& \ldots \\ du &=& \ldots\ dx \end{array}\right. & & \begin{array}{rcl} dv &=& \ldots\ dx \\ v &=& \ldots \end{array} \\
 & \quad\Downarrow \ \text{(just rearrange)} & & \quad\Downarrow \ \text{(just rearrange)} \\
(4)- & \left[\begin{array}{rcl} u &=& \ldots \\ du &=& \ldots\ dx \end{array}\right. & & \begin{array}{rcl} dv &=& \ldots\ dx \\ v &=& \ldots \end{array}
\end{array}
$$

Read the "diagonals" uv with alternating $\pm$ signs. So (1) is $+uv$, (2) is $-uv$, (3) is $+uv$, (4) is $-uv$, *etc.*

Read the "bottom horizontals" $v\,du$ as integrals, with the sign of the **next** level. So (1) becomes $-\int v\,du$, (2) is $+\int v\,du$, (3) is $-\int v\,du$, *etc.*

(Although you don't need these, the "top horizontals", $u\,dv$, are read as integrals with the same sign as their level, so (1) is $+\int u\,dv$, (2) is $-\int u\,dv$, *etc.* This way, the integration by parts formula $\int u\,dv = uv - \int v\,du$ is maintained.)

Within any level, calculate du and v by differentiating and integrating, respectively.

To go from one level to the next, either just move the dx over, leaving the rest of the expression unchanged (this is the usual step), **or** (if it makes things easier) you can rearrange other parts of the expression as well. The important thing is that both horizontal lines, $v\,du$ or $u\,dv$ must read the same, as algebraic expressions, after multiplying the parts, $v \times du$ and $u \times dv$, to get $v\,du$ and $u\,dv$. As long as this condition is met, you won't go wrong.

At any stage, the table can be read to give an equation for the original integral: read the diagonals (with their $\pm$) until finally reading a bottom $\int v\,du$ horizontal to finish. **N.B.** If a du is ever 0, then this last step also gives a 0, and so the integral is completely evaluated. So, stop the table if you get $du = 0$ at any level.

Note: This method can be streamlined, dropping the extra notation with u's and v's, and dropping the repeated horizontal lines from one level to the next. I leave that to you, once you get used to the principles involved. I think for most beginners, the presentation above, although a bit repetitive, is less likely to cause confusion, leading to errors.

2 Examples

Try these "longhand" without the table also, to see just how the table reflects the calculations you need to make, but simplifies keeping track of brackets, $\pm$ signs, and so on. You will then be better able to use the table with confidence (and to avoid silly errors!). The table doesn't replace "thinking" — it just helps keep track of repetitive calculations.

1. $$\int x^3 e^{2x}\, dx = \frac{1}{2}x^3 e^{2x} - \frac{3}{4}x^2 e^{2x} + \frac{6}{8}x\, e^{2x} - \frac{6}{16} e^{2x} + C$$

Our first example is perhaps the simplest and most typical use of the table, to handle repeated integration by parts. Since the table ends with a 0, we get the answer directly as shown.

$$\begin{array}{ll} (1)+ & \left[\begin{array}{rclrcl} u &=& x^3 & dv &=& e^{2x}\, dx \\ du &=& 3x^2\, dx & v &=& \frac{1}{2} e^{2x} \end{array}\right. \\ (2)- & \left[\begin{array}{rclrcl} u &=& 3x^2 & dv &=& \frac{1}{2} e^{2x}\, dx \\ du &=& 6x\, dx & v &=& \frac{1}{4} e^{2x} \end{array}\right. \\ (3)+ & \left[\begin{array}{rclrcl} u &=& 6x & dv &=& \frac{1}{4} e^{2x}\, dx \\ du &=& 6\, dx & v &=& \frac{1}{8} e^{2x} \end{array}\right. \\ (4)- & \left[\begin{array}{rclrcl} u &=& 6 & dv &=& \frac{1}{8} e^{2x} \\ du &=& 0\, dx & v &=& \frac{1}{16} e^{2x} \end{array}\right. \end{array}$$

There is a small "trick" here: we need to use substitution to calculate the v as $\int e^{2x}\, dx = \frac{1}{2} e^{2x}$: let $t = 2x$ and $dt = 2\, dx$.

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2. $$\int \arcsin x\, dx$$

Start the table:

$$(1)+ \quad \left[\begin{array}{rclrcl} u &=& \arcsin x & dv &=& dx \\ du &=& \frac{1}{\sqrt{1-x^2}}\, dx & v &=& x \end{array}\right.$$

Note that continuing in the usual way, letting the next level be

$$u = \frac{1}{\sqrt{1-x^2}} \qquad dv = x\, dx$$

we will get more and more complicated entries. But if we look at the bottom horizontal $\int v\, du$, we see a simple substitution

$$\int \frac{x\, dx}{\sqrt{1-x^2}} \qquad \text{(Let } t = 1 - x^2\,,\ dt = -2x\, dx)$$

So we continue with that rather than with integration by parts. (This can be done within the table, in fact, but that isn't necessary, and so I will skip making that explicit.)

$$\begin{aligned} \int \arcsin x\, dx &= x \arcsin x - \int \frac{x\, dx}{\sqrt{1-x^2}} \\ &= x \arcsin x - \left(-\frac{1}{2} \int \frac{dt}{\sqrt{t}} \right) \\ &= x \arcsin x + \frac{1}{2} \int t^{-1/2}\, dt \\ &= x \arcsin x + \frac{1}{2}\frac{2}{1} t^{1/2} + C \\ &= x \arcsin x + \sqrt{1-x^2} + C \end{aligned}$$

3. $$\int e^{2x} \sin x \, dx$$

This example illustrates "recursion": after two table levels, essentially the same integral reappears, so we stop the table, read it as an equation, to be solved.

$$\begin{array}{ll} (1)+ & \left[\begin{array}{rcl} u &=& e^{2x} \\ du &=& 2\,e^{2x}\,dx \end{array}\right. \quad \begin{array}{rcl} dv &=& \sin x\,dx \\ v &=& -\cos x \end{array} \\ (2)- & \left[\begin{array}{rcl} u &=& 2\,e^{2x} \\ du &=& 4\,e^{2x}\,dx \end{array}\right. \quad \begin{array}{rcl} dv &=& -\cos x\,dx \\ v &=& -\sin x \end{array} \end{array}$$

Writing down the equation this gives, we have:

$$\begin{aligned} \int e^{2x} \sin x \, dx &= \left(-\,e^{2x}\cos x\right) - \left(-2\,e^{2x}\sin x\right) + \left(-\int 4\,e^{2x}\sin x\,dx\right) \\ &= -\,e^{2x}\cos x + 2\,e^{2x}\sin x - 4\int e^{2x}\sin x\,dx \\ \text{So } 5\int e^{2x}\sin x\,dx &= -\,e^{2x}\cos x + 2\,e^{2x}\sin x \\ \text{\& so } \int e^{2x}\sin x\,dx &= -\frac{1}{5}\,e^{2x}\cos x + \frac{2}{5}\,e^{2x}\sin x + C \end{aligned}$$

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4. We finish with an example to illustrate that "back-substitution" may also be done using integration by parts (though you may prefer to continue to use "back-substitution"). (This particular example might also be tried with trigonometric substitution — decide which method you prefer.)

$$\int x^3\sqrt{x^2+1}\,dx$$

$$\begin{array}{ll} (1)+ & \left[\begin{array}{rcl} u &=& \frac{1}{2}x^2 \\ du &=& x\,dx \end{array}\right. \quad \begin{array}{rcl} dv &=& 2x\sqrt{x^2+1}\,dx \\ v &=& \frac{2}{3}(x^2+1)^{3/2} \end{array} \\ (2)- & \left[\begin{array}{rcl} u &=& \frac{1}{3} \\ du &=& 0 \end{array}\right. \quad \begin{array}{rcl} dv &=& 2x(x^2+1)^{3/2}\,dx \\ v &=& \frac{2}{5}(x^2+1)^{5/2} \end{array} \end{array}$$

$$\begin{aligned} \text{So } \int x^3\sqrt{x^2+1}\,dx &= \frac{1}{2}\,x^2\,\frac{2}{3}\,(x^2+1)^{3/2} - \frac{1}{3}\,\frac{2}{5}\,(x^2+1)^{5/2} \\ &= \frac{1}{3}\,(x^2+1)^{3/2} - \frac{2}{15}\,(x^2+1)^{5/2} + C \end{aligned}$$

In level 1 we arranged the parts of the integral so that the dv could be easily integrated, using the substitution $t = x^2 + 1$, $dt = 2x\,dx$, making $dv = t^{1/2}\,dt$.

In going from level 1 to level 2, we arranged the parts of the horizontal line so that again dv could be easily integrated, using in fact the same substitution.

Ending with $du = 0$ stopped the process, giving the final answer.

3 Exercises

Some to try yourself: (Also do the exercises in your text.)

1. $\int x^2 \sin 2x\,dx$ 2. $\int x\sec^2 x\,dx$ 3. $\int x \,\mathrm{arcsec}\, x\,dx$ 4. $\int e^{3x}\cos 5x\,dx$ 5. $\int x\ln x\,dx$