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Calculus II (Maths 201–NYB)

Euler's Equation

This is a lovely application of power series, though the proof that these functions do actually equal their Maclaurin series must wait till the beginning of Cal III (or do some reading on your own—that proof is in your textbook in the section on series!).

First we recall the Maclaurin series for these functions:

$$\begin{aligned}
\mathrm{e}^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots + \frac{x^n}{n!} + \cdots \\
\sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \cdots \\
\cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots
\end{aligned}$$

Next, we add new numbers to the real numbers, particularly $i = \sqrt{-1}$ (meaning we add a new number i with the property that $i^2 = -1$). Then we add all the other new numbers that result from combining i with arbitrary real numbers, forming such new numbers as $-i$, $2+i$, $-\pi + \sqrt{124}\,i$ and so on: in general, we consider all combinations $a+bi$, where a and b are real numbers. This resulting system is called the "complex numbers" (not because they are particularly complicated, but because they are combinations of real numbers and "imaginary" numbers (multiples of i)). One property of the complex numbers is that all ordinary computations with them only produces other complex numbers—you do not have to add more "new" numbers to get such numbers as $\sqrt{i}, (5+\pi i)^{(\mathrm{e}+i)}, \sin(2-3i)$ and so on. Note that $\sqrt{-1}$ has two square roots, as expected: i and $-i$.

Note also that simple powers of i are really simple:

$$\begin{aligned}
&i^0 = 1, \quad i^1 = i, \quad i^2 = -1, \quad i^3 = -i, \\
&i^4 = 1, \quad i^5 = i, \quad i^6 = -1, \quad i^7 = -i, \\
&i^8 = 1, \quad \ldots \text{ (the cycle repeats forever).}
\end{aligned}$$

These all are simple consequences of the defining property $i^2 = -1$. For example $i^3 = i^2 \cdot i = (-1) \cdot i$, and so on; the others are similar.

So, in particular, we can calculate $i\sin(x), \cos(x)$ and e^{ix} and then add $i\sin(x) + \cos(x)$ and compare with e^{ix}:

$$\begin{aligned}
i\sin x &= ix - i\frac{x^3}{3!} + i\frac{x^5}{5!} - i\frac{x^7}{7!} + \cdots \\
\cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots \\
\mathrm{e}^{ix} &= 1 + ix - \frac{x^2}{2!} - i\frac{x^3}{3!} + \frac{x^4}{4!} + i\frac{x^5}{5!} - \frac{x^6}{6!} - i\frac{x^7}{7!} + \cdots
\end{aligned}$$

So: $\mathrm{e}^{ix} = \cos(x) + i\sin(x)$ (This accounts for the alternate notation for $\mathrm{e}^{ix} := \mathrm{cis}(x)$.)

The final step now is to let $x = \pi$, and notice that $\sin(\pi) = 0$ and $\cos\pi = -1$: $\mathrm{e}^{i\pi} = -1$, or (perhaps more elegantly?):

$$\mathrm{e}^{\pi i} + 1 = 0$$