Calculating $\zeta(2)$—over and over again!

We shall start with the integral $\int \sin^k x\,dx$. Using integration by parts, we can show that the following recursion equation is true (for all values of k):

$$\int \sin^k x\,dx = -\frac{1}{k}\sin^{k-1} x\cos x + \frac{k-1}{k}\int \sin^{k-2} x\,dx$$

Denote the definite integral $\int_0^{\pi/2} \sin^k x\,dx$ by $\mathcal{I}(k)$. Note that by the above $\mathcal{I}(k) = \frac{k-1}{k}\mathcal{I}(k-2)$. Using this recursion formula, we can show that the following equation is true (for all integers $n > 0$):

$$\mathcal{I}(2n+1) = \int_0^{\pi/2} \sin^{2n+1} x\,dx = \frac{2n}{2n+1}\cdot\frac{2n-2}{2n-1}\cdot\dots\cdot\frac{2}{3}\cdot 1 \tag{1}$$

Next consider the power series representation of $\arcsin x$:

$$\arcsin x \quad = \quad x + \sum_{n=1}^{\infty} \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots(2n)}\cdot\frac{x^{2n+1}}{2n+1} \tag{2}$$

obtained by integrating the binomial expansion (as we did in class!):

$$\frac{1}{\sqrt{1-t^2}} = 1 + \sum \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots(2n)} t^{2n}$$

Using the change of variables $x = \sin\theta$ and equation (1), we can show that

$$\int_0^1 \frac{x^{2n+1}}{\sqrt{1-x^2}}\,dx = \int_0^{\pi/2} \sin^{2n+1}\theta\,d\theta = \frac{2\cdot 4\cdot 6\cdots(2n)}{1\cdot 3\cdot 5\cdots(2n+1)} \tag{3}$$

and by direct (Cal II) integration that

$$\int_0^1 \frac{\arcsin x}{\sqrt{1-x^2}}\,dx = \frac{\pi^2}{8} \tag{4}$$

Using the infinite series (equation (2)) for $\arcsin x$, and equation (3), it follows that

$$\int_0^1 \frac{\arcsin x}{\sqrt{1-x^2}}\,dx = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots = \sum_{n=1}^{\infty}\frac{1}{(2n-1)^2} \tag{5}$$

This series is the "odd-numbered half" of the p-series ($p = 2$); since absolutely convergent series may be rearranged, we can in fact rearrange things to show that this is $\frac{3}{4}$ of the full series:

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = \sum_{n=1}^{\infty}\frac{1}{(2n-1)^2} + \sum_{n=1}^{\infty}\frac{1}{(2n)^2} \quad\text{and}\quad \sum_{n=1}^{\infty}\frac{1}{(2n)^2} = \frac{1}{4}\sum_{n=1}^{\infty}\frac{1}{n^2}$$

and hence

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{3}{4} \sum_{n=1}^{\infty} \frac{1}{n^2} \quad \text{i.e.} \quad \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{4}{3} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \tag{6}$$

So we conclude using equations (4, 5, 6) that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{4}{3}\frac{\pi^2}{8} = \frac{\pi^2}{6}$$

This is a famous result of Euler's ("E as in e"). It was the first result obtained in summing p series $\sum \frac{1}{n^p}$, usually[1] denoted $\zeta(p)$. It is not too difficult (Euler did it!) to extend Euler's result to obtain formulas for all the even powers $\zeta(2n)$ (apparently he knew such formulas in the 18th century, although formal proofs for the formulas for $\zeta(2n)$ were not generally understood until later in the 19th century), but to this day, no formula is known for any of the odd powers, not even for the "simplest" $\zeta(3) = \sum \frac{1}{n^3}$. About 25 years ago $\zeta(3)$ was shown to be irrational, but beyond that little is known in terms of actual formulas like the one shown here for $\zeta(2)$.

[1] This "zeta function" $\zeta(s)$ is one of the really famous functions of mathematics, and a conjecture concerning its behaviour (the "Riemann hypothesis") is one of several million dollar problems that challenge mathematicians. You can find out more at `http://www.claymath.org/prizeproblems/`

Next we'll see two other proofs of the Euler formula using double integrals. They are "simple" in different ways; you decide which you think is simpler overall!

First:

We start by calculating $\int_0^1 \int_0^1 \frac{dx\,dy}{1-x^2y^2}$, an improper integral, but we shall ignore that for now (exercise: check the appropriate limit to show this does converge).

$$
\begin{aligned}
(1-x^2y^2)^{-1} &= 1 + x^2y^2 + \frac{(-1)(-2)}{2!}(-x^2y^2)^2 + \frac{(-1)(-2)(-3)}{3!}(-x^2y^2)^3 + \cdots \\
&= 1 + x^2y^2 + x^4y^4 + x^6y^6 + \cdots
\end{aligned}
$$

So

$$
\begin{aligned}
\int_0^1 \frac{dx}{1-x^2y^2} &= x + \frac{1}{3}x^3y^2 + \frac{1}{5}x^5y^4 + \frac{1}{7}x^7y^6 + \cdots \Big]_0^1 \\
&= 1 + \frac{1}{3}y^2 + \frac{1}{5}y^4 + \frac{1}{7}y^6 + \cdots
\end{aligned}
$$

So

$$
\begin{aligned}
\int_0^1 \int_0^1 \frac{dx}{1-x^2y^2}\,dy &= y + \frac{1}{3^2}y^3 + \frac{1}{5^2}y^5 + \frac{1}{7^2}y^7 + \cdots \Big]_0^1 \\
&= 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots \\
&= \frac{3}{4}\zeta(2)
\end{aligned}
$$

(by equation 6). In other words,

$$\zeta(2) = \frac{4}{3}\int_0^1 \int_0^1 \frac{dx\,dy}{1-x^2y^2}$$

Now we shall calculate this double integral another way, using the transformation

$$x = \frac{\sin(u)}{\cos(v)},\ y = \frac{\sin(v)}{\cos(u)}$$

over the triangle $\mathcal{T} = \{\langle u, v\rangle \mid u, v \geq 0, u + v \leq \pi/2\}$.

Note that this transformation maps $\mathcal{T}$ to the unit square $[0,1]\times[0,1]$ in the xy plane. Its Jacobian is $1-x^2y^2$, and the area of $\mathcal{T}$ is $\frac{1}{2}\times$ base $\times$ height $= \pi^2/8$.

So

$$\int_0^1 \int_0^1 \frac{dx\,dy}{1-x^2y^2} = \iint_{\mathcal{T}} du\,dv = \frac{\pi^2}{8}$$

and hence

$$\zeta(2) = \frac{4}{3}\frac{\pi^2}{8} = \frac{\pi^2}{6}$$

Second:

This time we calculate $\int_0^1 \int_0^1 \frac{dx\,dy}{1-xy}$, another improper integral (again, you should check the appropriate limit to show this also converges).

Again we use infinite series:

$$
\begin{aligned}
(1-xy)^{-1} &= 1 + xy + \frac{(-1)(-2)}{2!}(-xy)^2 + \frac{(-1)(-2)(-3)}{3!}(-xy)^3 + \cdots \\
&= 1 + xy + x^2y^2 + x^3y^3 + \cdots
\end{aligned}
$$

So

$$
\begin{aligned}
\int_0^1 \frac{dx}{1-xy} &= x + \frac{1}{2}x^2y + \frac{1}{3}x^3y^3 + \frac{1}{4}x^4y^3 + \cdots\Big]_0^1 \\
&= 1 + \frac{1}{2}y + \frac{1}{3}y^2 + \frac{1}{4}y^3 + \cdots
\end{aligned}
$$

So

$$
\begin{aligned}
\int_0^1 \int_0^1 \frac{dx}{1-xy}\,dy &= y + \frac{1}{2^2}y^2 + \frac{1}{3^2}y^3 + \frac{1}{4^2}y^4 + \cdots\Big]_0^1 \\
&= 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots
\end{aligned}
$$

In other words,

$$
\int_0^1 \int_0^1 \frac{dx\,dy}{1-xy} = \sum_{n=1}^{\infty} \frac{1}{n^2} = \zeta(2)
$$

We now evaluate the double integral another way, to obtain an actual value for $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

Effectively we shall rotate the unit square (and double its area) with the transformation $x = u - v$, $y = u + v$. First, as an exercise, you should show that this transformation takes the square $\mathcal{R}$: $[0,1] \times [0,1]$ to the diamond $\mathcal{S}$ given by these four lines: $v = -u$, $v = u$, $v = u - 1$, $v = 1 - u$. Furthermore, the Jacobian $\frac{\partial(x,y)}{\partial(u,v)} = 2$.

So we have the following calculation (there are hints below, so you can fill in the details for yourself).

$$
\begin{aligned}
\iint_{\mathcal{R}} \frac{1}{1-xy}\,dx\,dy &= 2\iint_{\mathcal{S}} \frac{1}{1-(u^2-v^2)}\,du\,dv \\
&= 2\int_0^{1/2}\int_{-u}^{u} \frac{1}{1-u^2+v^2}\,dv\,du + 2\int_{1/2}^{1}\int_{u-1}^{1-u} \frac{1}{1-u^2+v^2}\,dv\,du \\
&= 2\arcsin^2(\tfrac{1}{2}) - 2\arcsin^2(0) + \pi(\arcsin(1) - \arcsin(\tfrac{1}{2})) \\
&\qquad - (\arcsin^2(1) - \arcsin^2(\tfrac{1}{2})) \\
&= \frac{\pi^2}{18} - 0 + \frac{\pi^2}{2} - \frac{\pi^2}{6} - \frac{\pi^2}{4} + \frac{\pi^2}{36} \\
&= \frac{\pi^2}{6}
\end{aligned}
$$

Here are the relevant hints:

- $$\int \frac{dv}{a^2+v^2} = \frac{1}{a}\arctan\left(\frac{v}{a}\right)$$

  So $\displaystyle\int \frac{1}{1-u^2+v^2}\,dv = \frac{1}{\sqrt{1-u^2}}\arctan\left(\frac{v}{\sqrt{1-u^2}}\right)$.

- $$\arctan\left(\frac{u}{\sqrt{1-u^2}}\right) = \arcsin(u)$$

  So $\displaystyle\int \frac{1}{\sqrt{1-u^2}}\arctan\left(\frac{u}{\sqrt{1-u^2}}\right)du = \int \frac{\arcsin(u)}{\sqrt{1-u^2}}\,du = \frac{1}{2}(\arcsin(u))^2$.

- $$\arctan\left(\frac{1-u}{\sqrt{1-u^2}}\right) = \frac{1}{2}\arccos(u) = \frac{\pi}{4} - \frac{1}{2}\arcsin(u)$$

  So $\displaystyle\int \frac{1}{\sqrt{1-u^2}}\arctan\left(\frac{1-u}{\sqrt{1-u^2}}\right)du = \frac{\pi}{4}\int \frac{du}{\sqrt{1-u^2}} - \frac{1}{2}\int \frac{\arcsin(u)}{\sqrt{1-u^2}}\,du = \frac{\pi}{4}\arcsin(u) - \frac{1}{4}(\arcsin(u))^2$.

- And finally $\arcsin(x) = -\arcsin(-x)$ and $\arctan(x) = -\arctan(-x)$ (so we can "double up" the integrals of the form $\int_{-\alpha}^{\alpha}$ to get $2\int_0^{\alpha}$).