Chapter 5

Vector Geometry

In this chapter we will look more closely at certain geometric aspects of vectors in $\mathbb{R}^n$. We will first develop an intuitive understanding of some basic concepts by looking at vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$ where visualization is easy, then we will extend these geometric intuitions to $\mathbb{R}^n$ for any n. The basic geometric concepts that we will look at involve measurable quantities such as length, angle, area and volume. We also take a closer look at the two main types of equations covered in this course: parametric-vector equations and linear equations.

We begin with a reminder. We defined a vector in $\mathbb{R}^n$ as an n-tuple, i.e., as an $n \times 1$ matrix. This is an *algebraic* definition of a vector where a vector is just a list of numbers. The geometric objects we will look at in this chapter should be seen as *geometric interpretations* of this algebraic definition. One difficulty that students encounter at this stage is that there are many different geometric interpretations that can be given to a vector. For example, a vector in $\mathbb{R}^n$ can be interpreted geometrically as

- an arrow starting at the origin.
- an arrow with a certain length and direction but no fixed location.
- a point (or more exactly, the coordinates of a point relative to some reference point).
- a directed line segment between two points.
- a displacement (i.e., a translation).

This multiplicity of interpretations is a strength of the vector concept not a weakness. Vectors have many applications and depending on the application one geometric interpretation may be more relevant than another but no matter what geometric interpretation is chosen the underlying vector algebra remains the same. We will interpret a vector in $\mathbb{R}^n$ as a **position vector** as described in section 1.3 of Lay's textbook. A position vector is just a pointer to a certain location in $\mathbb{R}^n$. When using position vectors it is not necessary to make a firm distinction between a vector and its endpoint. For example, when we say that a line is a set of vectors we mean that the endpoints of the vectors lie on the line. If we want to stress the direction of the vector we will usually represent it as an arrow. If we want to stress the particular location that the vector is pointing to we will usually represent it by a point.

EXAMPLE 5.1. *If $A = (x_1, x_2, \ldots, x_n)$ and $B = (y_1, y_2, \ldots, y_n)$ are two points then* **the vector from** A **to** B *(represented by $\overrightarrow{AB}$) is defined as follows*

$$\overrightarrow{AB} = \begin{bmatrix} y_1 - x_1 \\ y_2 - x_2 \\ \vdots \\ y_n - x_n \end{bmatrix}$$

You can think of this as letting A be the origin of a new coordinate system and then the entries in $\overrightarrow{AB}$ give the location of B relative to A. Or you can imagine translating both A and B by subtracting A from both points so that A is translated to the origin. Finally, you can think of $\overrightarrow{AB}$ as an arrow from A to B.

So, for example, if we have $P(1, 5, 2)$ and $Q(7, 7, 0)$ then $\overrightarrow{PQ} = \begin{bmatrix} 7-1 \\ 7-5 \\ 0-2 \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \\ -2 \end{bmatrix}$. The entries in this vector indicate that when you travel from P to Q you move 6 units in the x_1 direction, 2 units in the x_2 direction and 2 units in the negative x_3 direction. These entries express the location of Q relative to P. If $\overrightarrow{PQ}$ is drawn with the *intial point at the origin then the terminal point would be (6, 2, -2).*

We will usually represent a vector as an $n \times 1$ matrix but there is another standard way of representing vectors that is frequently used. In $\mathbb{R}^2$ we define

$$\mathbf{i} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \mathbf{j} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

It then follows that any vector in $\mathbb{R}^2$ can be written as

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ b \end{bmatrix} = a\mathbf{i} + b\mathbf{j}$$

Similarly in $\mathbb{R}^3$ we define

$$\mathbf{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad \mathbf{j} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad \mathbf{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

and then any vector in $\mathbb{R}^3$ can be written

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$$

You should realize that in $\mathbb{R}^2$ the vectors $\mathbf{i}$ and $\mathbf{j}$ are just the vectors which we have called $\mathbf{e}_1$ and $\mathbf{e}_2$, the standard basis of $\mathbb{R}^2$. Similarly in $\mathbb{R}^3$ the vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are the standard basis of $\mathbb{R}^3$.

5.1 Distance and Length

The first geometric concept we want to look at is the the length of a vector. We define this to be the usual Euclidean distance from the intial point (the origin) to the end point of the vector. The length any vector $\mathbf{v}$ in $\mathbb{R}^n$ will be represented by $\|\mathbf{v}\|$. This quantity is also referred to as the *magnitude* or *norm* of $\mathbf{v}$.

Let $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ be a vector in $\mathbb{R}^2$. The length of this vector would be the distance from the origin $(0, 0)$ to the point (u_1, u_2) and this is given by the Pythagorean Theorem as

$$\|\mathbf{u}\| = \sqrt{u_1^2 + u_2^2}$$

EXAMPLE 5.2. *Let* $\mathbf{u} = \begin{bmatrix} 5 \\ -3 \end{bmatrix}$. *Figure 5.1 shows* $\mathbf{u}$ *and by the Pythagorean Theorem we can find the norm of* $\mathbf{u}$ *as*

$$\|\mathbf{u}\| = \sqrt{5^2 + (-3)^2} = \sqrt{34}$$

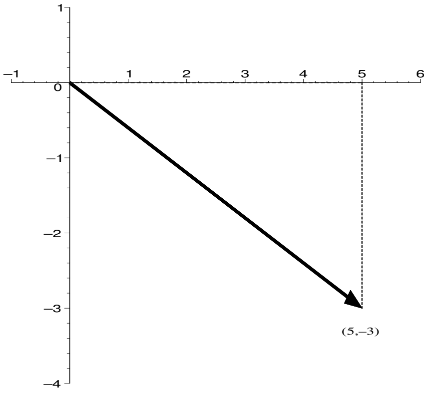

Figure 5.1.

In $\mathbb{R}^3$ a similar argument based on the Pythagorean Theorem gives

$$\|\mathbf{u}\| = \sqrt{u_1^2 + u_2^2 + u_3^2}$$

for any vector $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$.

We can extend the above formulas to $\mathbb{R}^n$ by defining

$$\|\mathbf{u}\| = \sqrt{u_1^2 + u_2^2 + \cdots + u_n^2}$$

Notice that if $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$ is any vector in $\mathbb{R}^n$ then

$$\mathbf{u}^T\mathbf{u} = \begin{bmatrix} u_1 & u_2 & \cdots & u_n \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} = u_1^2 + u_2^2 + \cdots + u_n^2$$

We then have the following concise formula which is valid for vectors in $\mathbb{R}^n$ for all n

$$\|\mathbf{u}\|^2 = \mathbf{u}^T\mathbf{u}$$

EXAMPLE 5.3. *Let* $\mathbf{u}$ *be any vector in* $\mathbb{R}^n$ *and* k *be a scalar then*

$$\begin{aligned} \|k\mathbf{u}\|^2 &= \left(k\mathbf{u}^T\right)(k\mathbf{u}) \\ &= k^2\mathbf{u}^T\mathbf{u} \\ &= k^2\|\mathbf{u}\|^2 \end{aligned}$$

Taking square roots then gives

$$\|k\mathbf{u}\| = |k|\,\|\mathbf{u}\|$$

This shows that multiplying any vector in $\mathbb{R}^n$ by a scalar k scales the length of the vector by $|k|$. We will sometimes make a distinction between the **sense** *of a vector and the* **direction** *of a vector. When a vector is multiplied by a negative scalar the reversal of the arrow is described by saying the sense has been reversed but the direction has stayed the same.*

Definition 5.1. *The* **distance** *between two vectors* $\mathbf{u}$ *and* $\mathbf{v}$ *in* $\mathbb{R}^n$ *is defined as* $\|\mathbf{u} - \mathbf{v}\|$.

EXAMPLE 5.4. *The distance between* $\mathbf{u} = \mathbf{i} + \mathbf{k}$ *and* $\mathbf{v} = \mathbf{j} - \mathbf{k}$ *is*

$$\|\mathbf{u} - \mathbf{v}\| = \|\mathbf{i} - \mathbf{j} + 2\mathbf{k}\| = \sqrt{1^2 + (-1)^2 + 2^2} = \sqrt{6}$$

Unit Vectors

A **unit vector** is a vector whose length is 1.

If $\mathbf{u}$ is any non-zero vector in $\mathbb{R}^n$ then $\dfrac{1}{\|\mathbf{u}\|}\mathbf{u}$ is a unit vector. This can be seen by applying the formula $\|\mathbf{v}\|^2 = \mathbf{v}^T\mathbf{v}$ to the vector $\dfrac{1}{\|\mathbf{u}\|}\mathbf{u}$. This gives:

$$\begin{aligned}\left(\frac{1}{\|\mathbf{u}\|}\mathbf{u}\right)^T\left(\frac{1}{\|\mathbf{u}\|}\mathbf{u}\right) &= \frac{1}{\|\mathbf{u}\|^2}\mathbf{u}^T\mathbf{u}\\ &= \frac{1}{\|\mathbf{u}\|^2}\|\mathbf{u}\|^2\\ &= 1\end{aligned}$$

The process of multiplying a vector by the reciprocal of its length to obtain a unit vector is called **normalization**. Notice that this procedure doesn't alter the direction or sense of the vector.

EXAMPLE 5.5. *Normalize the vector* $\mathbf{v} = \begin{bmatrix} 2 \\ 2 \\ 0 \\ -1 \end{bmatrix}$.

We have $\|\mathbf{v}\| = \sqrt{4+4+0+1} = \sqrt{9} = 3$ *so*

$$\frac{1}{3}\begin{bmatrix} 2 \\ 2 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 2/3 \\ 2/3 \\ 0 \\ -1/3 \end{bmatrix}$$

is a unit vector parallel to $\mathbf{v}$*. Note: Just to avoid any possible confusion, when we say that two non-zero vectors,* $\mathbf{u}$ *and* $\mathbf{v}$*, are parallel we mean that they have the same direction. Each one is a scalar multiple of the other.*

Problems

1. If $A = (4, -2)$ and $\overrightarrow{AB} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$ what is B?
2. If $B = (5, -4, 7)$ and $\overrightarrow{AB} = \begin{bmatrix} -6 \\ 2 \\ 2 \end{bmatrix}$ what is A?
3. Find the length of the following vectors:

   a. $\begin{bmatrix} 3 \\ -2 \end{bmatrix}$

   b. $\begin{bmatrix} 1 \\ 4 \\ -1 \end{bmatrix}$

   c. $\begin{bmatrix} 4 \\ 3 \\ 2 \\ 1 \end{bmatrix}$

   d. $\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$

   e. $\begin{bmatrix} \cos(s)\sin(t) \\ \cos(s)\cos(t) \\ \sin(s) \end{bmatrix}$

   f. $\mathbf{i} + \mathbf{j} + \mathbf{k}$

   g. $4\mathbf{i} - \mathbf{j} - 3\mathbf{k}$

   h. $\sqrt{1-2t^2}\mathbf{i} + t\mathbf{j} + t\mathbf{k}$

4. Let $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$ be the vector in $\mathbb{R}^n$ all of whose entries are 1. What is $\|\mathbf{v}\|$?
5. Find the lengths of the sides of triangle ABC where the vertices are given by

   a. $A(0,0), B(3,3), C(5,-1)$

   b. $A(-1,2), B(1,5), C(3,1)$

   c. $A(1,0,0)$, $B(0,1,0)$, $C(0,0,1)$

   d. $A(3,1,2)$, $B(4,-1,-2)$, $C(-2,0,1)$

6. a. If $\overrightarrow{AB} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, $\overrightarrow{BC} = \begin{bmatrix} -3 \\ 5 \end{bmatrix}$, and A is the point $(3,7)$ what is C? Draw a diagram illustating this problem.

   b. If $\overrightarrow{PQ} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$, $\overrightarrow{QR} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$, and R is the point $(-3,5,2)$ what is P?
7. Let $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} k \\ k+1 \\ k+2 \end{bmatrix}$. Use calculus to find the value of k for which the distance from $\mathbf{u}$ to $\mathbf{v}$ is a minimum.
8. Find a unit vector parallel to each of the following vectors:

   a. $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$

   b. $\begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$

   c. $\begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}$

   d. $3\mathbf{i} - 5\mathbf{j} + 2\mathbf{k}$

   e. $\begin{bmatrix} 1 \\ t \end{bmatrix}$

   f. $\begin{bmatrix} \cos t + \sin t \\ \cos t - \sin t \end{bmatrix}$

9. If $\|\overrightarrow{AB}\| = 5$ and $\|\overrightarrow{BC}\| = 3$ what are the possible values for $\|\overrightarrow{AC}\|$?
10. Let $\mathbf{u} = \begin{bmatrix} \cos(s) \\ \sin(s) \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix}$. These are two unit vectors in $\mathbb{R}^2$. Show that the distance from $\mathbf{u}$ to $\mathbf{v}$ is $\sqrt{2 - 2\cos(s-t)}$
11. Prove that in $\mathbb{R}^3$ the length of $\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ is given by $\sqrt{v_1^2 + v_2^2 + v_3^2}$.
12. True or False:

    a. $\|\mathbf{u}\|^2 = \mathbf{u}\mathbf{u}^T$

    b. $\|\mathbf{u}\|^2 = \mathbf{u}^T\mathbf{u}$

    c. $\|2\mathbf{u}\|^2 = 4\mathbf{u}^T\mathbf{u}$

    d. $\|\mathbf{u} + \mathbf{v}\|^2 = \mathbf{u}^T\mathbf{u} + \mathbf{v}^T\mathbf{v}$

    e. If $\|\mathbf{u}\| = \|\mathbf{v}\|$ then $\|\mathbf{u} + \mathbf{v}\| = \|\mathbf{u}\| + \|\mathbf{v}\|$.

    f. $\|A\mathbf{u}\|^2 = \mathbf{u}^T A^T A\mathbf{u}$
13. Under what conditions will $\|\mathbf{u} + \mathbf{v}\| = \|\mathbf{u}\| + \|\mathbf{v}\|$?
14. Suppose A is an $n \times n$ matrix such that $A^T A = I$. Let $\mathbf{v}$ be any vector in $\mathbb{R}^n$. Show $\|A\mathbf{v}\| = \|\mathbf{v}\|$
15. Let $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$. Show that if $\mathbf{v}$ is any vector in $\mathbb{R}^2$ then $\|A\mathbf{v}\| = \|\mathbf{v}\|$.

5.2 The Dot Product

The Dot Product in $\mathbb{R}^2$

Suppose we have two vectors in $\mathbb{R}^2$, $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ as illustrated in Figure 5.2

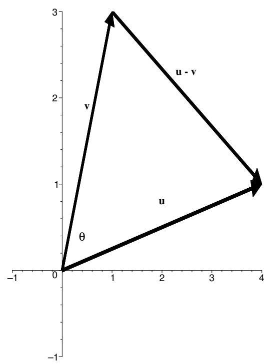

Figure 5.2.

These two vectors determine a triangle whose third side would be $\mathbf{u} - \mathbf{v}$ translated. If we let θ be the angle between $\mathbf{u}$ and $\mathbf{v}$ then we can apply the law of cosines to the triangle. This gives

$$\begin{aligned} \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2\|\mathbf{u}\|\,\|\mathbf{v}\|\cos\theta &= \|\mathbf{u} - \mathbf{v}\|^2 \\ &= (u_1 - v_1)^2 + (u_2 - v_2)^2 \\ &= \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2u_1u_2 - 2v_1v_2 \end{aligned}$$

Cancelling out common factors and terms we get

$$\|\mathbf{u}\|\,\|\mathbf{v}\|\cos\theta = u_1v_1 + u_2v_2$$

The expression on the right hand side of the last line is given a special name. It is called the **dot product** of $\mathbf{u}$ and $\mathbf{v}$ and is written $\mathbf{u}\cdot\mathbf{v}$. Thus we have the following two formulas

$$\mathbf{u}\cdot\mathbf{v} = u_1v_1 + u_2v_2$$

and

$$\mathbf{u}\cdot\mathbf{v} = \|\mathbf{u}\|\,\|\mathbf{v}\|\cos\theta$$

where θ is the angle between $\mathbf{u}$ and $\mathbf{v}$. Since θ is one angle of a triangle we have $0^\circ \leq \theta \leq 180^\circ$. This means that θ is the smallest positive angle between $\mathbf{u}$ and $\mathbf{v}$.

There is another way of indicating the dot product. If $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ then

$$\mathbf{u}^T\mathbf{v} = \begin{bmatrix} u_1 & u_2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = u_1v_1 + u_2v_2$$

and so we have the nice formula

$$\mathbf{u}\cdot\mathbf{v} = \mathbf{u}^T\mathbf{v}$$

(This formula uses the standard convention of interpreting the 1×1 matrix $\mathbf{u}^T\mathbf{v}$ as a scalar.)

EXAMPLE 5.6. *If* $\mathbf{u} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ *and* $\mathbf{v} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ *then*

$$\mathbf{u}\cdot\mathbf{v} = (1)(3) + (3)(1) = 6$$

and

$$\|\mathbf{u}\| = \|\mathbf{v}\| = \sqrt{10}$$

It then follows that if θ *is the angle between* $\mathbf{u}$ *and* $\mathbf{v}$ *we have*

$$\cos\theta = \frac{6}{\sqrt{10}\sqrt{10}} = \frac{3}{5}$$

and the angle between $\mathbf{u}$ *and* $\mathbf{v}$ *is* $\arccos(3/5) \approx 51.13^\circ$.

The dot product is often useful in geometric problems involving angles. If the problem is stated in terms of points then it should be "translated" into vector terminology before using the dot product. Look at the following example:

EXAMPLE 5.7. *Draw the parabola* $y = x^2$ *and let* P *be the point* $(1,1)$ *on this parabola. If* O *is the origin find another point* Q *on the parabola such that the angle between* OP *and* OQ *is* 30°.

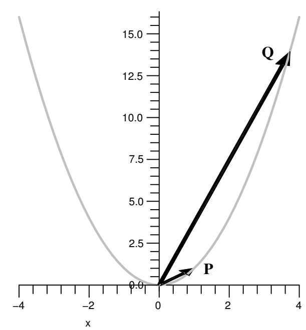

Figure 5.3.

In terms of vectors we have $\overrightarrow{OP} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ *and* $\overrightarrow{OQ} = \begin{bmatrix} x \\ x^2 \end{bmatrix}$. *We also have* $\overrightarrow{OP}\cdot\overrightarrow{OQ} = x + x^2$, $\|\overrightarrow{OP}\| = \sqrt{2}$, *and*

$\|\overrightarrow{OQ}\| = \sqrt{x^2+x^4}$. *If the angle between these vectors is* $30°$ *the dot product formula gives*

$$x + x^2 = \sqrt{2}\sqrt{x^2+x^4}\cos 30°$$

Substituting $\cos 30° = \frac{\sqrt{3}}{2}$ *and squaring both sides gives*

$$x^2 + 2x^3 + x^4 = 2\left(x^2+x^4\right)\frac{3}{4}$$

We omit the algebra but you should be able to solve this equation and find two values of x *that work* $x = 2+\sqrt{3}$ *and* $x = 2-\sqrt{3}$.

The Dot Product in $\mathbb{R}^n$

The dot product can be generalized to vectors in $\mathbb{R}^n$.

Definition 5.2. *Let* $\mathbf{u}$ *and* $\mathbf{v}$ *be vectors in* $\mathbb{R}^n$ *then their dot product is defined by*

$$\mathbf{u}\cdot\mathbf{v} = \mathbf{u}^T\mathbf{v}$$

The above definition implies that if $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ then

$$\mathbf{u}\cdot\mathbf{v} = u_1v_1 + u_2v_2 + \cdots + u_nv_n$$

The fundamental properties of the dot product are summarized by the following theorem.

Theorem 5.3. *If* $\mathbf{x}$, $\mathbf{y}$, *and* $\mathbf{z}$ *are vectors in* $\mathbb{R}^n$ *and* c *is a scalar then*

a. $\mathbf{x}\cdot\mathbf{y} = \mathbf{y}\cdot\mathbf{x}$

b. $\mathbf{x}\cdot(\mathbf{y}+\mathbf{z}) = \mathbf{x}\cdot\mathbf{y} + \mathbf{x}\cdot\mathbf{z}$

c. $(c\mathbf{x})\cdot\mathbf{y} = c\,(\mathbf{x}\cdot\mathbf{y})$

d. $\mathbf{x}\cdot\mathbf{x} \geq 0$ *and* $\mathbf{x}\cdot\mathbf{x} = 0$ *if and only if* $\mathbf{x} = \mathbf{0}$.

Proof. The proof of part (a) is left as an exercise.

Since $\mathbf{u}\cdot\mathbf{v} = \mathbf{u}^T\mathbf{v}$ parts (b) and (c) follow from the corresponding properties of matrix multiplication.

The proof of part (d) is more complicated because it makes several claims. First it claims that the inner product of a vector with itself can never be negative. To see this let $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$. Then $\mathbf{x}\cdot\mathbf{x} = x_1^2 + x_2^2 + \cdots + x_n^2$ and this value can never be negative since it is the sum of squares.

Next it says that the inner product of the zero vector with itself is 0. This is easy: if $\mathbf{x} = \mathbf{0}$ then $\mathbf{x}\cdot\mathbf{x} = 0+0+\cdots+0 = 0$.

Finally part (d) claims that if the dot product of a vector with itself is 0 then the vector must be the zero vector. To see this suppose $\mathbf{x}\cdot\mathbf{x} = 0$. Then $x_1^2 + x_2^2 + \cdots + x_n^2 = 0$. Since the left hand side of this equation has no negative terms the only way the terms can add up to 0 is if each term is 0. So $x_1 = 0$, $x_2 = 0$, …, $x_n = 0$ and we have $\mathbf{x} = \mathbf{0}$.

This last property says that in $\mathbb{R}^n$ the only vector with length 0 is the zero vector.

□

Orthogonal Vectors

The formula $\mathbf{u}\cdot\mathbf{v} = \|\mathbf{u}\|\,\|\mathbf{v}\|\cos\theta$ relates the dot product to the angle between vectors [1]. If $\mathbf{u}$ and $\mathbf{v}$ are non-zero vectors then the right hand side of this expression is positive if $0° \leq \theta < 90°$ and negative if $90° < \theta \leq 180°$. More importantly the dot product is 0 if $\theta = 90°$. This means that two non-zero vectors are perpendicular if and only if their dot product is 0. This leads to the following definition.

Definition 5.4. *Two vectors in* $\mathbb{R}^n$ *are said to be* **orthogonal** *if their dot product is 0.*

This definition implies that the zero vector in $\mathbb{R}^n$ is orthogonal to every vector in $\mathbb{R}^n$.

EXAMPLE 5.8. *Show that the triangle with vertices* $A(-2,3)$, $B(5,5)$, *and* $C(0,-4)$ *is a right triangle.*

We want to interpret the sides of this triangle as vectors. If we treat A *as the origin then the vector from* A *to* B *would be* $\overrightarrow{AB} = \begin{bmatrix} 5-(-2) \\ 5-3 \end{bmatrix} = \begin{bmatrix} 7 \\ 2 \end{bmatrix}$. *Similarly* $\overrightarrow{AC} = \begin{bmatrix} 0-(-2) \\ -4-3 \end{bmatrix} = \begin{bmatrix} 2 \\ -7 \end{bmatrix}$. *Now notice that* $\overrightarrow{AB}\cdot\overrightarrow{AC} = 7(2)+2(-7) = 0$. *Since* $\overrightarrow{AB}\cdot\overrightarrow{AC} = 0$ *we have* $\|\overrightarrow{AB}\|\,\|\overrightarrow{AC}\|\cos\theta = 0$. *We can then conclude that*

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[1]

Although this formula was proved only for vectors in $\mathbb{R}^2$ it is applicable in all $\mathbb{R}^n$. Justification for this will be given shortly.

$\cos\theta = 0$ and so $\theta = 90°$.

EXAMPLE 5.9. *For what value(s) of k are the vectors $\mathbf{u} = \begin{bmatrix} k \\ 2k \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} k+1 \\ k-1 \end{bmatrix}$ orthogonal?*

We just have to determine any values of k which make $\mathbf{u}\cdot\mathbf{v} = 0$.

$$\begin{aligned} \mathbf{u}\cdot\mathbf{v} &= k(k+1) + 2k(k-1) \\ &= k^2 + k + 2k^2 - 2k \\ &= 3k^2 - k \\ &= k(3k-1) \end{aligned}$$

It should then be clear that the vectors are orthogonal for $k = 0$ and $k = 1/3$.

We end this section with three theorems which state some important properties of vectors.

Theorem 5.5. *(**Pythagorean Theorem**) Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in $\mathbb{R}^n$ then $\|\mathbf{u}+\mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$ if and only if $\mathbf{u}$ and $\mathbf{v}$ are orthogonal.*

Proof. Note that the statement of this theorem is another "if and only if" statement. This means that the theorem is making two claims. These will be proved separately below.

First we must show that if $\mathbf{u}$ and $\mathbf{v}$ are orthogonal then $\|\mathbf{u}+\mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$. The argument is as follows

$$\begin{aligned} \|\mathbf{u}+\mathbf{v}\|^2 &= (\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}+\mathbf{v}) && (5.1) \\ &= \mathbf{u}\cdot\mathbf{u} + 2\mathbf{u}\cdot\mathbf{v} + \mathbf{v}\cdot\mathbf{v} && (5.2) \\ &= \mathbf{u}\cdot\mathbf{u} + 0 + \mathbf{v}\cdot\mathbf{v} && (5.3) \\ &= \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 && (5.4) \end{aligned}$$

Equation 5.2 is a consequence of the distributive and commutative properties of the dot product. Equation 5.3 is a consequence of $\mathbf{u}$ and $\mathbf{v}$ being orthogonal (their dot product is 0).

Next we must show that if $\|\mathbf{u}+\mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$ then $\mathbf{u}$ and $\mathbf{v}$ are orthogonal. The argument goes as follows

$$\begin{aligned} \|\mathbf{u}+\mathbf{v}\|^2 &= \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 && (5.5) \\ \mathbf{u}\cdot\mathbf{u} + 2\mathbf{u}\cdot\mathbf{v} + \mathbf{v}\cdot\mathbf{v} &= \mathbf{u}\cdot\mathbf{u} + \mathbf{v}\cdot\mathbf{v} && (5.6) \\ 2\mathbf{u}\cdot\mathbf{v} &= 0 && (5.7) \\ \mathbf{u}\cdot\mathbf{v} &= 0 && (5.8) \end{aligned}$$

The last line tells us that $\mathbf{u}$ and $\mathbf{v}$ are orthogonal. □

Theorem 5.6. *(**Cauchy-Schwarz Theorem**) If $\mathbf{u}$ and $\mathbf{v}$ are vectors in $\mathbb{R}^n$ then*

$$|\mathbf{u}\cdot\mathbf{v}| \le \|\mathbf{u}\|\|\mathbf{v}\|$$

Proof. If $\mathbf{v}$ is the zero vector then both sides of the inequality would be zero and the theorem would be true so we will assume that $\mathbf{v}$ is not the zero vector.

Part d of **Theorem 5.3** tells us that

$$\left(\mathbf{u} - \frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\mathbf{v}\right)\cdot\left(\mathbf{u} - \frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\mathbf{v}\right) \ge 0$$

since the dot product of any vector with itself is always greater than or equal to 0.

Simplifying the left hand side of the above gives

$$\begin{aligned} \left(\mathbf{u} - \frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\mathbf{v}\right)\cdot\left(\mathbf{u} - \frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\mathbf{v}\right) &= \mathbf{u}\cdot\mathbf{u} - 2\frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\mathbf{u}\cdot\mathbf{v} + \frac{(\mathbf{u}\cdot\mathbf{v})^2}{(\mathbf{v}\cdot\mathbf{v})^2}\mathbf{v}\cdot\mathbf{v} \\ &= \mathbf{u}\cdot\mathbf{u} - \frac{(\mathbf{u}\cdot\mathbf{v})^2}{\mathbf{v}\cdot\mathbf{v}} \end{aligned}$$

Replacing this last expression in the original inequality gives

$$\|\mathbf{u}\|^2 - \frac{(\mathbf{u}\cdot\mathbf{v})^2}{\|\mathbf{v}\|^2} \ge 0$$

Rearranging these terms gives

$$\|\mathbf{u}\|^2\,\|\mathbf{v}\|^2 \ge (\mathbf{u}\cdot\mathbf{v})^2$$

Taking the square root of both sides gives

$$|\mathbf{u}\cdot\mathbf{v}| \le \|\mathbf{u}\|\|\mathbf{v}\|$$

□

The Cauchy-Schwarz Theorem guarantees that

$$-1 \le \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|} \le 1$$

for any non-zero vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$. This allows us to define the angle θ between any non-zero vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$ by the formula

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|}$$

(where $0 \le \theta \le 180°$).

EXAMPLE 5.10. *Find the angle between $\mathbf{u} = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$.*

We have $\mathbf{u}\cdot\mathbf{v} = 1 - 2 + 3 - 4 = -2$, $\|\mathbf{u}\| = \sqrt{1+1+1+1} = \sqrt{4} = 2$, and $\|\mathbf{v}\| = \sqrt{1+4+9+16} = \sqrt{30}$. Thus

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|} = \frac{-2}{2\sqrt{30}} \approx -.1826$$

which gives $\theta \approx 100.52°$

Theorem 5.7. *(* **Triangle Inequality***) If* $\mathbf{u}$ *and* $\mathbf{v}$ *are vectors in* $\mathbb{R}^n$ *then*

$$\|\mathbf{u}+\mathbf{v}\| \le \|\mathbf{u}\| + \|\mathbf{v}\|$$

Proof.

$$\begin{aligned}
\|\mathbf{u}+\mathbf{v}\|^2 &= \mathbf{u}\cdot\mathbf{u} + 2\mathbf{u}\cdot\mathbf{v} + \mathbf{v}\cdot\mathbf{v} && (5.9)\\
&\le \mathbf{u}\cdot\mathbf{u} + 2|\mathbf{u}\cdot\mathbf{v}| + \mathbf{v}\cdot\mathbf{v} && (5.10)\\
&\le \mathbf{u}\cdot\mathbf{u} + 2\|\mathbf{u}\|\|\mathbf{v}\| + \mathbf{v}\cdot\mathbf{v} && (5.11)\\
&= (\|\mathbf{u}\| + \|\mathbf{v}\|)^2 && (5.12)
\end{aligned}$$

Equation 5.11 follows from the Cauchy-Schwarz theorem

The above shows that $\|\mathbf{u}+\mathbf{v}\|^2 \le (\|\mathbf{u}\|+\|\mathbf{v}\|)^2$. The theorem then follows by taking the square root of both sides. □

Direction Angles

The dot product gives us a new way of looking at unit vectors. Any particular entry in a unit vector cannot be larger than 1 or less than -1. The entries in a unit vector turn out to have a very simple geometric interpretation.

In $\mathbb{R}^3$ the angles between any vector $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ and the x_1, x_2 and x_3 axes are called the **direction angles** of $\mathbf{v}$ and are represented by α, β, γ respectively. These are just the angles between $\mathbf{v}$ and the unit vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$. So we have

$$\begin{aligned}
\cos\alpha &= \frac{\mathbf{v}\cdot\mathbf{i}}{\|\mathbf{v}\|\,\|\mathbf{i}\|} = \frac{v_1}{\|\mathbf{v}\|}\\
\cos\beta &= \frac{\mathbf{v}\cdot\mathbf{j}}{\|\mathbf{v}\|\,\|\mathbf{j}\|} = \frac{v_2}{\|\mathbf{v}\|}\\
\cos\gamma &= \frac{\mathbf{v}\cdot\mathbf{k}}{\|\mathbf{v}\|\,\|\mathbf{k}\|} = \frac{v_3}{\|\mathbf{v}\|}
\end{aligned}$$

We can now write

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} \|\mathbf{v}\|\cos\alpha \\ \|\mathbf{v}\|\cos\beta \\ \|\mathbf{v}\|\cos\gamma \end{bmatrix} = \|\mathbf{v}\| \begin{bmatrix} \cos\alpha \\ \cos\beta \\ \cos\gamma \end{bmatrix}$$

The values $\cos\alpha$, $\cos\beta$, and $\cos\gamma$ are called the **direction cosines** of $\mathbf{v}$.

The last example can be generalized in the following way (we leave the details as an exercise). Let $\mathbf{v}$ be any non-zero vector in $\mathbb{R}^n$. If we normalize $\mathbf{v}$ then the entries in the normalized vector are just the cosines of the angles between $\mathbf{v}$ and the vectors in the standard basis of $\mathbb{R}^n$. That is, the entries in any unit vector are the direction cosines of that vector.

EXAMPLE 5.11. *Let* $\mathbf{u} = \begin{bmatrix} 0 \\ 1 \\ -1 \\ \sqrt{2} \end{bmatrix}$. *If we normalize* $\mathbf{u}$ *we get* $\begin{bmatrix} 0 \\ 1/2 \\ -1/2 \\ \sqrt{2}/2 \end{bmatrix}$. *Now* $\cos^{-1}(0) = \pi/2$, $\cos^{-1}(1/2) = \pi/3$, $\cos^{-1}(-1/2) = 2\pi/3$, *and* $\cos^{-1}(\sqrt{2}/2) = \pi/4$. *So* $\mathbf{u}$ *lies at an angle of* $90°$ *from* $\mathbf{e}_1$, *at an angle of* $60°$ *from* $\mathbf{e}_2$, *at an angle of* $120°$ *from* $\mathbf{e}_3$, *at an angle of* $45°$ *from* $\mathbf{e}_4$

Note that for any non-zero vector $\mathbf{v}$ there are two unit vectors in the same direction as $\mathbf{v}$. One has the same sense as $\mathbf{v}$, the other has the opposite sanse of $\mathbf{v}$.

Problems

1. For the following pairs of vectors $\mathbf{u}$ and $\mathbf{v}$ calculate $\|\mathbf{u}\|$, $\|\mathbf{v}\|$, $\mathbf{u}\cdot\mathbf{v}$, and the angle between $\mathbf{u}$ and $\mathbf{v}$.

   a. $\mathbf{u} = \begin{bmatrix}2\\-1\end{bmatrix}$, $\mathbf{v} = \begin{bmatrix}3\\4\end{bmatrix}$

   b. $\mathbf{u} = \begin{bmatrix}1\\5\end{bmatrix}$, $\mathbf{v} = \begin{bmatrix}-3\\0\end{bmatrix}$

   c. $\mathbf{u} = \begin{bmatrix}1\\1\\1\end{bmatrix}$, $\mathbf{v} = \begin{bmatrix}1\\-2\\3\end{bmatrix}$

   d. $\mathbf{u} = \begin{bmatrix}2\\1\\2\\1\end{bmatrix}$, $\mathbf{v} = \begin{bmatrix}4\\3\\2\\1\end{bmatrix}$

   e. $\mathbf{u} = \begin{bmatrix}a\\b\end{bmatrix}$, $\mathbf{v} = \begin{bmatrix}b\\a\end{bmatrix}$

   f. $\mathbf{u} = \mathbf{i}-\mathbf{j}+\mathbf{k}$, $\mathbf{v} = 2\mathbf{i}+\mathbf{j}+2\mathbf{k}$

2. Use the dot product to find the angles of the triangles with the following vertices

   a. $A(1,4)$, $B(3,-2)$, $C(6,1)$

   b. $A(1,0,1)$, $B(0,2,1)$, $C(2,1,0)$

   c. $A(1,1,2,2)$, $B(1,2,2,1)$, $C(2,2,1,1)$

3. Given the points $A(1,1)$, $B(3,-1)$, and $C(4,k)$ find the values of k for which triangle ABC is a right triangle.

4. Let $\mathbf{u}_1 = \begin{bmatrix}3\\1\\1\end{bmatrix}$ and $\mathbf{u}_2 = \begin{bmatrix}1\\0\\1\end{bmatrix}$. Show that any vector in Span $\{\mathbf{u}_1, \mathbf{u}_2\}$ is orthogonal to $\mathbf{v} = \begin{bmatrix}1\\-2\\-1\end{bmatrix}$.

5. Let $\mathbf{x} = \begin{bmatrix}a\\b\end{bmatrix}$ and $R = \begin{bmatrix}\cos(\theta) & -\sin(\theta)\\ \sin(\theta) & \cos(\theta)\end{bmatrix}$. Use the dot product to find the angle between $\mathbf{x}$ and $R\mathbf{x}$.

6. Let $\mathbf{u} = \begin{bmatrix}1\\3\end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix}2\\k\end{bmatrix}$. For what value(s) of k

   a. do $\mathbf{u}$ and $\mathbf{v}$ have the same length?

   b. are $\mathbf{u}$ and $\mathbf{v}$ orthogonal?

   c. are $\mathbf{u}$ and $\mathbf{v}$ parallel?

   d. is the distance from $\mathbf{u}$ to $\mathbf{v}$ one unit?

7. Let $\mathbf{u} = \begin{bmatrix}1\\2\\3\end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix}2\\1+k\\1-k\end{bmatrix}$. For what value(s) of k

   a. do $\mathbf{u}$ and $\mathbf{v}$ have the same length?

   b. are $\mathbf{u}$ and $\mathbf{v}$ orthogonal?

   c. are $\mathbf{u}$ and $\mathbf{v}$ parallel?

   d. is the distance from $\mathbf{u}$ to $\mathbf{v}$ 3 units?

8. Let $\mathbf{u} = \begin{bmatrix}\cos\theta\\ \sin\theta\end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix}\sin\theta\\ \cos\theta\end{bmatrix}$. For what value(s) of θ

   a. do $\mathbf{u}$ and $\mathbf{v}$ have the same length?

   b. are $\mathbf{u}$ and $\mathbf{v}$ orthogonal?

   c. are $\mathbf{u}$ and $\mathbf{v}$ parallel?

9. Normalize the following vectors and find the direction angles in each case:

   a. $\mathbf{v} = \begin{bmatrix}1\\2\\-2\end{bmatrix}$ b. $\mathbf{v} = \begin{bmatrix}-3\\1\\4\end{bmatrix}$ c. $\mathbf{v} = \begin{bmatrix}1\\2\\3\\4\\5\end{bmatrix}$

10. a. Suppose $\mathbf{v}$ is a vector in $\mathbb{R}^2$ and you know that this vector forms an angle of $\pi/3$ with $\mathbf{i}$. Is this enough information to determine vector $\mathbf{v}$? What are the possible values for the angle between $\mathbf{v}$ and $\mathbf{j}$?

    b. Suppose $\mathbf{v}$ is a vector in $\mathbb{R}^3$ which forms an angle of $\pi/6$ with $\mathbf{i}$. What are the possible values for the angle between $\mathbf{v}$ and $\mathbf{k}$?

    c. Suppose $\mathbf{v}$ is a vector in $\mathbb{R}^3$ which forms an angle of $\pi/6$ with $\mathbf{i}$ and an angle of $\pi/6$ with $\mathbf{j}$. What are the possible values for the angle between $\mathbf{v}$ and $\mathbf{k}$?

11. Use the dot product to prove that
$$\|\mathbf{x}+\mathbf{y}\|^2 + \|\mathbf{x}-\mathbf{y}\|^2 = 2\left(\|\mathbf{x}\|^2 + \|\mathbf{y}\|^2\right)$$

12. Show that if $\mathbf{u}+\mathbf{v}$ and $\mathbf{u}-\mathbf{v}$ have the same magnitude then $\mathbf{u}$ and $\mathbf{v}$ are orthogonal.

13. It was pointed out in this section that $\mathbf{u}^T\mathbf{v} = \mathbf{v}^T\mathbf{u}$ for any two vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$. Is it also true that $\mathbf{u}\mathbf{v}^T = \mathbf{v}\mathbf{u}^T$? Prove this or give a counterexample.

14. Suppose $\mathbf{x}$ and $\mathbf{y}$ are vectors in $\mathbb{R}^n$ that have the same length. Show that $\mathbf{x}+\mathbf{y}$ bisects the angle between $\mathbf{x}$ and $\mathbf{y}$.

15. Prove part a of **Theorem 4.3**.

16. Let $\mathbf{u}$ and $\mathbf{v}$ be two vectors in $\mathbb{R}^n$.

    a. Justify the following steps
$$\mathbf{u}^T\mathbf{v} = (\mathbf{u}^T\mathbf{v})^T = \mathbf{v}^T\mathbf{u}$$

    b. The following chain of equalities resembles the above example but is not valid. What is wrong with the following?
$$\mathbf{u}\mathbf{v}^T = (\mathbf{u}\mathbf{v}^T)^T = \mathbf{v}\mathbf{u}^T$$

5.3 Lines

Lines in $\mathbb{R}^2$

We have already seen in Chapter 1 of Lay's textbook that a line in $\mathbb{R}^2$ can be represented by an equation of the form $\mathbf{x} = \mathbf{x}_0 + t\mathbf{v}$. Such an equation represents the line through $\mathbf{x}_0$ parallel to $\mathbf{v}$. This is called a parametric-vector equation of the line. We now want to look at another way of representing a line in $\mathbb{R}^2$.

How can we specify the direction of a line in $\mathbb{R}^2$? One standard way is to give the slope of the line and another is to give a vector parallel to the line. There is a third way of specifying the direction of a line in $\mathbb{R}^2$ which is not so obvious and which at first might seem unnecessarily complicated. We can specify the direction by giving a vector perpendicular to the line. Such a vector is said to be a **normal** vector to the line.

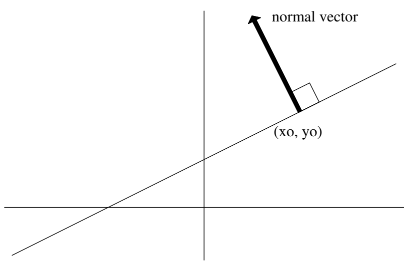

Figure 5.4.

Suppose $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$ lies on the straight line through $\mathbf{x_0} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$ with normal $\mathbf{n} = \begin{bmatrix} a \\ b \end{bmatrix}$ then $\mathbf{x} - \mathbf{x_0}$ is parallel to the line and is therefore perpendicular to $\mathbf{n}$. We must then have

$$\mathbf{n} \cdot (\mathbf{x} - \mathbf{x_0}) = 0$$

$$\begin{bmatrix} a \\ b \end{bmatrix} \cdot \begin{bmatrix} x - x_0 \\ y - y_0 \end{bmatrix} = 0$$

$$a(x - x_0) + b(y - y_0) = 0$$

$$ax + by - ax_0 - by_0 = 0$$

The equation of the line can then be written as

$$ax + by = ax_0 + by_0$$

or more concisely

$$\boxed{\text{ax+by=c}}$$

where $c = ax_0 + by_0$. We will call this the **normal** equation of a straight line.[2] Notice that this equation could be written in vector notation as

$$\boxed{\mathbf{n}^T\mathbf{x} = \mathbf{n}^T\mathbf{x_0}}$$

Now you already knew that any straight line in $\mathbb{R}^2$ could be written in the form $ax + by = c$ but what is new here is that in an equation of this form the coefficients of x and y give a normal vector to the line.

EXAMPLE 5.12. *Given the parametric-vector equation of a line* $\mathbf{x} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} + t\begin{bmatrix} 4 \\ 3 \end{bmatrix}$ *find the normal equation of this line.*

We will use the following observation: If $\mathbf{u} = \begin{bmatrix} a \\ b \end{bmatrix}$ *is any vector in* $\mathbb{R}^2$ *then* $\begin{bmatrix} b \\ -a \end{bmatrix}$ *is orthogonal to* $\mathbf{u}$. *This can be easily confirmed by a simple dot product.*

$$\begin{bmatrix} a \\ b \end{bmatrix} \cdot \begin{bmatrix} b \\ -a \end{bmatrix} = ab - ab = 0$$

Now it should be clear that the direction of the given line is determined by the vector $\begin{bmatrix} 4 \\ 3 \end{bmatrix}$. *We want a normal vector to this line and the above comments show that* $\mathbf{n} = \begin{bmatrix} 3 \\ -4 \end{bmatrix}$ *would be such a vector. We now know that the normal equation of the line is* $3x - 4y = c$ *with the value of* c *yet to be determined.*

Now we use the fact that $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ *lies on the line and so these values must satisfy the equation. We then have* $3(3) - 4(1) = c$. *So* $c = 9 - 4 = 5$ *and the equation we are looking for is*

$$3x - 4y = 5$$

EXAMPLE 5.13. *Find an equation for the line through* $A(3,5)$ *and* $B(6,1)$.

The vector $\overrightarrow{AB} = \begin{bmatrix} 6-3 \\ 1-5 \end{bmatrix} = \begin{bmatrix} 3 \\ -4 \end{bmatrix}$ *is a direction vector for this line. The line* $\mathbf{x} = t\begin{bmatrix} 3 \\ -4 \end{bmatrix}$ *would be a line through the origin parallel to the desired line. We only have to add a translation. A parametric-vector equation for this line would be*

$$\mathbf{x} = t\begin{bmatrix} 3 \\ -4 \end{bmatrix} + \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

This is illustrated in Figure 5.5.

2

This is not the usual term for this equation. Most books would call it the **standard equation** or the **Cartesian equation** of a line. We are calling it the normal equation just to stress the fact that the coefficients give a normal vector to the line.

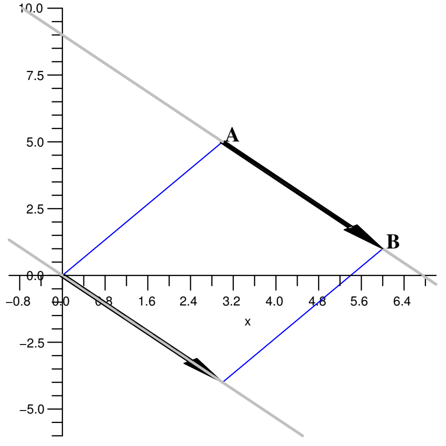

Figure 5.5. *The line through A and B.*

But we can also find a normal vector to this line, $\mathbf{n} = \begin{bmatrix}4\\3\end{bmatrix}$. *The line therefore has an equation of the form* $4x + 3y = c$. *Substituting the coordinates of A gives*

$$4x + 3y = 27$$

Lines in $\mathbb{R}^n$

In general a line in $\mathbb{R}^n$ can be written in parametric-vector form $\mathbf{x} = \mathbf{u} + t\mathbf{v}$ where $\mathbf{v}$ determines the direction of the line and $\mathbf{u}$ lies on the line.

EXAMPLE 5.14. *Find the equation of the line containing both* $\mathbf{p} = \begin{bmatrix}1\\2\\3\\4\end{bmatrix}$ *and* $\mathbf{q} = \begin{bmatrix}4\\3\\2\\1\end{bmatrix}$.

If we translate this line to pass through the origin by subtracting $\mathbf{p}$ *from all points on the line then a direction vector for the line would be*

$$\mathbf{q} - \mathbf{p} = \begin{bmatrix}4\\3\\2\\1\end{bmatrix} - \begin{bmatrix}1\\2\\3\\4\end{bmatrix} = \begin{bmatrix}3\\1\\-1\\-3\end{bmatrix}$$

The line $\mathbf{x} = t\begin{bmatrix}3\\1\\-1\\-3\end{bmatrix}$ *would therefore be a line through the origin parallel to the line we are looking for. We just have to translate the line so that it passes through* $\mathbf{p}$ *and* $\mathbf{q}$ *and we do this by adding either one of these to our equation. So one possible answer would be*

$$\mathbf{x} = \begin{bmatrix}1\\2\\3\\4\end{bmatrix} + t\begin{bmatrix}3\\1\\-1\\-3\end{bmatrix}$$

You should understand that points on the line are generated by different values of the parameter t. *In particular when* $t = 0$ *in the above equation we have* $\mathbf{x} = \mathbf{p}$ *and when* $t = 1$ *we have* $\mathbf{x} = \mathbf{q}$.

In $\mathbb{R}^n$ with $n \geq 3$ a vector that is orthogonal to a line can be pointing in infinitely many different directions. For example in $\mathbb{R}^3$ any vector in the x, y plane will be orthogonal to the z axis. This means that it is no longer possible to determine the direction of a line just by specifying a normal vector in these spaces.

So parametric equations are more versatile since they can be used to represent lines in spaces of any dimension. Lines can be represented by normal or Cartesian equations only in $\mathbb{R}^2$.

EXAMPLE 5.15. *Let* $\mathcal{L}_1$ *be the line defined by*

$$\mathbf{x} = \begin{bmatrix}1\\2\\1\\3\end{bmatrix} + s\begin{bmatrix}2\\1\\1\\3\end{bmatrix}$$

and $\mathcal{L}_2$ *be the line defined by*

$$\mathbf{x} = \begin{bmatrix}1\\2\\1\\3\end{bmatrix} + t\begin{bmatrix}3\\0\\2\\-1\end{bmatrix}$$

It should be obvious that these lines intersect at $\begin{bmatrix}1\\2\\1\\3\end{bmatrix}$. *The angle between the direction vectors of these lines is* $\cos^{-1}\frac{6+0+2-3}{\sqrt{4+1+1+9}\sqrt{9+0+4+1}} = \cos^{-1}\frac{5}{\sqrt{15}\sqrt{14}} \approx 1.22$ *radians. This is the angle formed by the two intersecting lines. Even though these are lines in* $\mathbb{R}^4$ *there is still a plane (a flat 2-dimensional surface) containing these lines. It should also be obvious that this plane is defined by the equation*

$$\mathbf{x} = \begin{bmatrix}1\\2\\1\\3\end{bmatrix} + s\begin{bmatrix}2\\1\\1\\3\end{bmatrix} + t\begin{bmatrix}3\\0\\2\\-1\end{bmatrix}$$

Does $\begin{bmatrix} 32 \\ 4 \\ 21 \\ 0 \end{bmatrix}$ *lie in this plane? This is equivalent to asking if the equation*

$$\begin{bmatrix} 1 \\ 2 \\ 1 \\ 3 \end{bmatrix} + s \begin{bmatrix} 2 \\ 1 \\ 1 \\ 3 \end{bmatrix} + t \begin{bmatrix} 3 \\ 0 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 32 \\ 4 \\ 21 \\ 0 \end{bmatrix}$$

has a solution. This vector equation gives us the augmented matrix

$$\begin{bmatrix} 2 & 3 & 31 \\ 1 & 0 & 2 \\ 1 & 2 & 20 \\ 3 & -1 & -3 \end{bmatrix}$$

This augmented matrix reduces to

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 9 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

So this point does lie in the plane. It corresponds to the values $s = 2$ *and* $t = 9$ *in the equation of the plane.*

Problems

1. Write the following lines in parametric-vector form.

   a. $3x_1 + x_2 = 5$
   b. $2x_1 - 5x_2 = 1$
   c. $x_1 = 4$
   d. $x_2 = -3$

2. Write the following lines in normal form

   a. $\mathbf{x} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} + t \begin{bmatrix} 3 \\ -1 \end{bmatrix}$
   b. $\mathbf{x} = \begin{bmatrix} 0 \\ 3 \end{bmatrix} + t \begin{bmatrix} 5 \\ 2 \end{bmatrix}$
   c. $\mathbf{x} = t \begin{bmatrix} 4 \\ 3 \end{bmatrix}$
   d. $\mathbf{x} = t\mathbf{i} + (1-t)\mathbf{j}$

3. Given the line $\mathcal{L}: \quad \mathbf{x} = \begin{bmatrix} -3 \\ 1 \end{bmatrix} + t \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

   a. Find the value of $\mathbf{x}$ that corresponds to $t = 1$. Find the value of $\mathbf{x}$ that corresponds to $t = -2$.
   b. Find the value of t that corresponds to $\mathbf{x} = \begin{bmatrix} 1 \\ -7 \end{bmatrix}$.
   c. Find all $\mathbf{x}$ on $\mathcal{L}$ that lie 2 units from $\begin{bmatrix} -3 \\ 1 \end{bmatrix}$.
   d. Find all $\mathbf{x}$ on $\mathcal{L}$ that lie 5 units from the origin.
   e. Illustrate all the above with a picture.

4. Given the line $\mathbf{x} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

   a. Find the value of $\mathbf{x}$ that correspond to $t = 1$. Find the values of $\mathbf{x}$ that correspond to $t = -2$.
   b. Find the value of t that corresponds to $\mathbf{x} = \begin{bmatrix} 4/3 \\ 2/3 \\ 1/3 \end{bmatrix}$.
   c. Is $\begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}$ on this line? For what value(s) of k is $\begin{bmatrix} 0 \\ 2 \\ k \end{bmatrix}$ on this line?
   d. Find all $\mathbf{x}$ on the line that lie 2 units from $\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$.
   e. Find all $\mathbf{x}$ on the line that lie 2 units from the origin.

5. Find all values of a and b such that the following two equations represent the same line

$$\mathbf{x} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} + t \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$
$$\mathbf{x} = \begin{bmatrix} a \\ 2 \end{bmatrix} + t \begin{bmatrix} b \\ 4 \end{bmatrix}$$

6. Find all values of a and b such that the following two equations represent the same line

$$\mathbf{x} = \begin{bmatrix} 4 \\ -2 \end{bmatrix} + t \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$
$$\mathbf{x} = \begin{bmatrix} a \\ 2 \end{bmatrix} + t \begin{bmatrix} b \\ 4 \end{bmatrix}$$

7. Find all values of a and b such that the following two equations represent the same line

$$4x_1 + x_2 = 3$$
$$2x_1 + ax_2 = b$$

8. Let $\mathbf{n}$ and $\mathbf{x}_0$ be vectors in $\mathbb{R}^2$. Show that the line through $\mathbf{x}_0$ normal to $\mathbf{n}$ can be written $\mathbf{n}^T\mathbf{x} = \mathbf{n}^T\mathbf{x}_0$.

9. Find a parametric vector equation of the line containing $\mathbf{p}$ and $\mathbf{q}$ where

   a. $\mathbf{p} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad \mathbf{q} = \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix}$
   b. $\mathbf{p} = \begin{bmatrix} 0 \\ 2 \\ 1 \\ -3 \end{bmatrix} \quad \mathbf{q} = \begin{bmatrix} 2 \\ 2 \\ 5 \\ 5 \end{bmatrix}$

10. In $\mathbb{R}^3$ find three different lines through the origin orthogonal to the vector $\mathbf{k}$.

11. For what values of a and b do the following equations give the same line?

$$\mathbf{x} = \begin{bmatrix} a \\ 3 \\ a+3 \end{bmatrix} + t \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \quad \text{and} \quad \mathbf{x} = s \begin{bmatrix} b \\ 2 \\ b+2 \end{bmatrix}$$

12. a. Show that the line in $\mathbb{R}^n$ which contains $\mathbf{p}$ and $\mathbf{q}$ can be represented by the equation $\mathbf{x} = (1-t)\mathbf{p} + t\mathbf{q}$.
    b. Find the normal equation of the line $\mathbf{x} = (1-t) \begin{bmatrix} 2 \\ -5 \end{bmatrix} + t \begin{bmatrix} 1 \\ 3 \end{bmatrix}$.

5.4 Planes and Hyperplanes

Planes in $\mathbb{R}^3$.

We have already seen in Chapter 1 of Lay's textbook that a plane in $\mathbb{R}^3$ can be expressed in parametric-vector form as

$$\mathbf{x} = \mathbf{x_0} + s\mathbf{u} + t\mathbf{v}$$

The expression $s\mathbf{u} + t\mathbf{v}$ corresponds to a plane through the origin generated by vectors $\mathbf{u}$ and $\mathbf{v}$ (also called the **span** of $\mathbf{u}$ and $\mathbf{v}$). The addition of $\mathbf{x_0}$ to this expression translates this plane in $\mathbb{R}^3$.

Now a plane in $\mathbb{R}^3$ doesn't have a slope and a plane doesn't point in any particular direction but a plane does have an **orientation** and parallel planes have the same orientation. The orientation of a plane in $\mathbb{R}^3$ can be specified by giving two independent vectors parallel to the plane or by specifying one vector that is normal to the plane. Suppose we want to determine a condition on $\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ that tells us when it is in the plane containing $\mathbf{x_0} = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix}$ with normal vector $\mathbf{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$. If both $\mathbf{x}$ and $\mathbf{x_0}$ lie in the plane then the vector $\mathbf{x} - \mathbf{x_0}$ must be parallel to the plane and so we have

$$\begin{aligned} \mathbf{n} \cdot (\mathbf{x} - \mathbf{x_0}) &= 0 \\ \begin{bmatrix} a \\ b \\ c \end{bmatrix} \cdot \begin{bmatrix} x - x_0 \\ y - y_0 \\ z - z_0 \end{bmatrix} &= 0 \\ a(x - x_0) + b(y - y_0) + c(z - z_0) &= 0 \\ ax + by + cz - ax_0 - by_0 - cz_0 &= 0 \end{aligned}$$

This equation can be written $ax + by + cz = ax_0 + by_0 + cz_0$ or just

$$\boxed{ax + by + cz = d}$$

where $d = ax_0 + by_0 + cz_0$. We will call this the **normal equation** [3] of a plane in $\mathbb{R}^3$. Note that this equation can be expressed in terms of the relevant vectors as

$$\boxed{\mathbf{n}^T\mathbf{x} = \mathbf{n}^T\mathbf{x_0}}$$

EXAMPLE 5.16. *The equation $x + 2y + 4z = 0$ can be expressed in the form $\mathbf{n}^T\mathbf{x} = \mathbf{n}^T\mathbf{x_0}$ as*

$$\begin{bmatrix} 1 & 2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 2 & 4 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

It is the equation of a plane with normal vector $\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$ which contains the point $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. A plot of the plane and normal vector is shown in Figure 5.6.

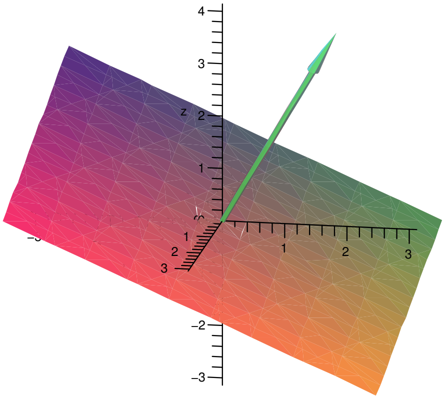

Figure 5.6. *The plane $x + 2y + 4z = 0$ and a normal vector.*

EXAMPLE 5.17. *Find a parametric-vector equation of the plane with normal equation*

$$x - 2y + 2z = 4$$

We can look at this equation as a very simple system of equations. It is just a system of one equation with three unknowns. The general solution will therefore require two free variables. The general solution will be

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 + 2s - 2t \\ s \\ t \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$$

and this is a parametric-vector equation of the plane.

Notice that the equation $x - 2y + 2z = 4$ tell us that $\mathbf{n} = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$ is a normal vector to the plane. The parametric-

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[3] It is more common to call this the **standard equation** or the **Cartesian equation** of a plane.

vector equation tells us that the plane is a translation of the plane generated by the vectors $\begin{bmatrix}2\\1\\0\end{bmatrix}$ *and* $\begin{bmatrix}-2\\0\\1\end{bmatrix}$. *A couple of simple dot product calculations shows us that* $\mathbf{n}$ *is perpendicular to both of these vectors as we should expect.*

EXAMPLE 5.18. *Find the normal equation of the plane containing* $\mathbf{p} = \begin{bmatrix}1\\1\\0\end{bmatrix}$, $\mathbf{q} = \begin{bmatrix}1\\0\\1\end{bmatrix}$ *and* $\mathbf{r} = \begin{bmatrix}0\\1\\1\end{bmatrix}$.

The equation that we are looking for has the form $ax + by + cz = d$. *Since the plane contains* $\mathbf{p}$ *we know that* $x = 1$, $y = 1$ *and* $z = 0$ *must satisfy this equation. So we have* $a(1) + b(1) + c(0) = d$. *Repeating this procedure for* $\mathbf{q}$ *and* $\mathbf{r}$ *results in the system*

$$\begin{aligned} a + b - d &= 0 \\ a + c - d &= 0 \\ b + c - d &= 0 \end{aligned}$$

This gives the the coefficient matrix

$$\begin{bmatrix}1 & 1 & 0 & -1\\ 1 & 0 & 1 & -1\\ 0 & 1 & 1 & -1\end{bmatrix}$$

which reduces to

$$\begin{bmatrix}1 & 0 & 0 & -1/2\\ 0 & 1 & 0 & -1/2\\ 0 & 0 & 1 & -1/2\end{bmatrix}$$

The general solution of this system is $\begin{bmatrix}a\\b\\c\\d\end{bmatrix} = t\begin{bmatrix}1/2\\1/2\\1/2\\1\end{bmatrix}$.

Choosing $t = 2$ *gives the particular solution* $a = b = c = 1$, $d = 2$ *and so one possible equation for the plane would be*

$$x + y + z = 2$$

(There are infinitely many solutions to this system which means there are infinitely many equations for this plane, but they are all just scalar multiples of the above equation.)

EXAMPLE 5.19. *Find the line of intersection of the planes* $x + y + 2z = 3$ *and* $x + 2y - z = 5$.

It should be clear that these planes are not parallel since their normals are not parallel. Since they are not parallel they should intersect along a straight line. The planes and the line of intersection are illustrated in Figure 5.7. The points of intersection are the points that satisfy both equations. So we want the solution of the system

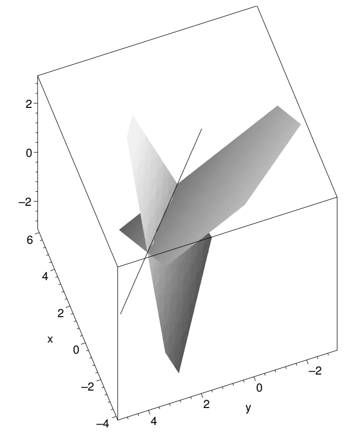


Figure 5.7. *Intersecting planes.*

$$\begin{aligned} x + y + 2z &= 3 \\ x + 2y - z &= 5 \end{aligned}$$

But this system can be written in the form $A\mathbf{x} = \mathbf{b}$ *where the solutions are now represented as vectors. The augmented matrix and its reduce form would be*

$$\begin{bmatrix}1 & 1 & 2 & 3\\ 1 & 2 & -1 & 5\end{bmatrix} \sim \begin{bmatrix}1 & 0 & 5 & 1\\ 0 & 1 & -3 & 2\end{bmatrix}$$

It should now be easy to write down the solution

$$\mathbf{x} = \begin{bmatrix}1-5t\\2+3t\\t\end{bmatrix} = \begin{bmatrix}1\\2\\0\end{bmatrix} + t\begin{bmatrix}-5\\3\\1\end{bmatrix}$$

This is a parametric vector representation of the line of intersection.

The direction vector of this line is $\begin{bmatrix}-5\\3\\1\end{bmatrix}$. *This vector must be parallel to both planes. Why? It then follows that this vector should be orthogonal to the normals for both planes. Verify that this is the case.*

EXAMPLE 5.20. *Find the point of intersection between the plane* $x + 2y - z = 3$ *and the line* $\mathbf{x} = \begin{bmatrix}1\\-5\\-3\end{bmatrix} + t\begin{bmatrix}1\\2\\2\end{bmatrix}$.

The equation of the line can be written as

$$\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}1\\-5\\-3\end{bmatrix} + t\begin{bmatrix}1\\2\\2\end{bmatrix} = \begin{bmatrix}1+t\\-5+2t\\-3+2t\end{bmatrix}$$

This means that any point on the line must satisfy the equations $x = 1 + t$, $y = -5 + 2t$, and $z = -3 + 2t$ for some value of the parameter t. If the point also lies on the plane it must satisfy the equation $x + 2y - z = 3$. Substitution then gives

$$\begin{aligned} x + 2y - z &= 3 \\ (1+t) + 2(-5+2t) - (-3+2t) &= 3 \\ 3t - 6 &= 3 \\ t &= 3 \end{aligned}$$

The point of intersection occurs when $t = 3$ so the point of intersection is $x = 1 + 3 = 4$, $y = -5 + 6 = 1$, $z = -3 + 6 = 3$, or in vector form $\mathbf{x} = \begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix}$.

Hyperplanes

We have seen that in $\mathbb{R}^2$ the equation $\mathbf{n}^T\mathbf{x} = \mathbf{n}^T\mathbf{x}_0$ defines a line through $\mathbf{x}_0$ with normal $\mathbf{n}$ and in $\mathbb{R}^3$ this equation defines a plane through $\mathbf{x}_0$ with normal $\mathbf{n}$.

In general, in $\mathbb{R}^n$ the equation $\mathbf{n}^T\mathbf{x} = \mathbf{n}^T\mathbf{x}_0$ defines what is called a **hyperplane** through $\mathbf{x}_0$ with normal $\mathbf{n}$. This type of equation is a linear equation with n variables. There are then $n-1$ free variables and this means that in $\mathbb{R}^n$ a hyperplane will be an $n-1$ dimensonal subspace that might or might not have been translated away from the origin. So in $\mathbb{R}^2$ a hyperplane is a (possibly translated) 1 dimensional subspace, a line. In $\mathbb{R}^3$ a hyperplane is a 2 dimensional subspace that has possibly been translated.

EXAMPLE 5.21. *Let $\mathbf{n} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$ and $\mathbf{x}_0 = \begin{bmatrix} 2 \\ 0 \\ 5 \\ 1 \end{bmatrix}$ then the normal equation of the hyperplane through $\mathbf{x}_0$ with normal $\mathbf{n}$ would be*

$$\begin{aligned} &\mathbf{n}^T\mathbf{x} = \mathbf{n}^T\mathbf{x_0} \\ &\begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 5 \\ 1 \end{bmatrix} \\ &x_1 + 2x_2 + 3x_3 + 4x_4 = 21 \end{aligned}$$

This equation contains four variables so the general solution would involve three free variables. If we represent the free variables by parameters $x_2 = r$, $x_3 = s$, and $x_4 = t$ then the general solution would be

$$\begin{aligned} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} &= \begin{bmatrix} 21 - 2r - 3s - 4t \\ r \\ s \\ t \end{bmatrix} \\ &= \begin{bmatrix} 21 \\ 0 \\ 0 \\ 0 \end{bmatrix} + r \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -4 \\ 0 \\ 0 \\ 1 \end{bmatrix} \end{aligned}$$

This would be a parametric-vector representation of the same hyperplane. This representation makes it clear that the hyperplane is a three dimensional subspace of $\mathbb{R}^4$ with basis

$$\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

that has been translated by the vector $\begin{bmatrix} 21 \\ 0 \\ 0 \\ 0 \end{bmatrix}$.

Another notation for lines and planes

A line through the origin in $\mathbb{R}^n$ is a one-dimensional subspace of $\mathbb{R}^n$. A plane through the origin in $\mathbb{R}^n$ is a two-dimensional subspace of $\mathbb{R}^n$. If we represent such a subspace by V then we can translate the line or plane to a new position by adding a vector to each vector in V. The translated line or plane can be represented by $V + \mathbf{u}$. In other words $V + \mathbf{u} = \{\mathbf{x} : \mathbf{x} = \mathbf{v} + \mathbf{u}, \mathbf{v} \in V\}$.

But there is no need to restrict ourselves to lines and planes. If V is any subspace of $\mathbb{R}^n$ then $V + \mathbf{u}$ represents V translated by $\mathbf{u}$ to a new position. If V is 1-dimensional then $V + \mathbf{u}$ is a line. If V is 2-dimensional then $V + \mathbf{u}$ is a plane. If V is $n - 1$ dimensional then $V + \mathbf{u}$ is a hyperplane. In general if V is k dimensional then $V + \mathbf{u}$ is called a **k-flat**.

Problems

1. Find a normal equation of the plane or hyperplane containing $\mathbf{x_0}$ with normal $\mathbf{n}$.

   a. $\mathbf{n} = \begin{bmatrix}1\\2\\-1\end{bmatrix}$, $\mathbf{x_0} = \begin{bmatrix}3\\0\\2\end{bmatrix}$

   b. $\mathbf{n} = \begin{bmatrix}3\\3\\0\end{bmatrix}$, $\mathbf{x_0} = \begin{bmatrix}1\\2\\5\end{bmatrix}$

   c. $\mathbf{n} = \mathbf{i} + \mathbf{j} + 3\mathbf{k}$, $\mathbf{x_0} = 2\mathbf{i} + 5\mathbf{j} - 3\mathbf{k}$

   d. $\mathbf{n} = \begin{bmatrix}3\\1\\2\\2\end{bmatrix}$, $\mathbf{x_0} = \begin{bmatrix}0\\2\\3\\1\end{bmatrix}$

2. Find a parametric-vector equation of each of the following planes or hyperplanes.

   a. $x_1 + x_2 - 2x_3 = 3$

   b. $3x_1 + 2x_2 + x_3 = 5$

   c. $x_1 + x_2 + x_3 + x_4 = 1$

3. The plane $x + 3y - z = 0$ is in fact a subspace of $\mathbb{R}^3$. Find a basis for this subspace and find an equation for the line through the origin normal to this plane.

4. The hyperplane $x_1 + 3x_2 - x_3 + 2x_4 = 0$ is in fact a subspace of $\mathbb{R}^4$. Find a basis for this subspace and find an equation for the line through the origin normal to this hyperplane.

5. Find an equation for the plane containing $\begin{bmatrix}3\\3\\2\end{bmatrix}$ parallel to the plane $x - 2y + 4z = 0$.

6. Find an equation for the line which containins $\begin{bmatrix}5\\0\\-1\end{bmatrix}$ and is also perpendicular to the plane $x + y - 4z = 3$.

7. The angle between two hyperplanes is defined to be the smallest possible angle between normals of the hyperplanes. This angle must be between 0 and $\pi/2$ radians (why?). This means that you should choose normals such that their dot product is positive. Find the angle between the following hyperplanes:

   a. $x + y + 3z = 2$, $2x - y - z = 4$

   b. $x_1 + 2x_2 + 3x_3 + 4x_4 = 0$, $4x_1 - 3x_2 - 2x_3 + x_4 = 1$

8. In $\mathbb{R}^n$ let $\mathbf{n} = \begin{bmatrix}1\\1\\\vdots\\1\end{bmatrix}$ and $\mathbf{x_0} = \begin{bmatrix}1\\1\\\vdots\\1\end{bmatrix}$. Find the normal equation and the parametric-vector equation of the hyperplane containing $\mathbf{x}_0$ with normal $\mathbf{n}$ for

   a. $\mathbb{R}^2$ b. $\mathbb{R}^3$ c. $\mathbb{R}^4$ d. $\mathbb{R}^5$

9. Find the point of intersection of the following planes and lines

   a. The plane $2x + 3y + 3z = 8$ and the line $\mathbf{x} = \begin{bmatrix}1\\4\\2\end{bmatrix} + t\begin{bmatrix}3\\0\\-1\end{bmatrix}$.

   b. The plane $3x - y - z = 5$ and the line $\mathbf{x} = \begin{bmatrix}1\\2\\5\end{bmatrix} + t\begin{bmatrix}2\\1\\5\end{bmatrix}$.

   c. The plane $x + y + 2z = 14$ and the line $\mathbf{x} = \begin{bmatrix}2\\3\\0\end{bmatrix} + t\begin{bmatrix}3\\1\\1\end{bmatrix}$.

   d. The plane $x + y + 2z = 14$ and the line $\mathbf{x} = \begin{bmatrix}3\\3\\4\end{bmatrix} + t\begin{bmatrix}1\\1\\-1\end{bmatrix}$

   e. The hyperplane $x_1 + x_2 - 2x_3 + x_4 = 3$ and the line $\mathbf{x} = \begin{bmatrix}8\\2\\-3\\3\end{bmatrix} + t\begin{bmatrix}3\\1\\-1\\2\end{bmatrix}$.

10. Let $\mathcal{P}$ be the plane $x + 3y + 2z = 9$ and let $\mathcal{L}$ be the line $\mathbf{x} = \begin{bmatrix}2\\1\\0\end{bmatrix} + t\begin{bmatrix}k\\k+2\\2\end{bmatrix}$. For what value(s) of k

    a. are $\mathcal{P}$ and $\mathcal{L}$ orthogonal?

    b. are $\mathcal{P}$ and $\mathcal{L}$ parallel?

    c. do $\mathcal{P}$ and $\mathcal{L}$ intersect at $\begin{bmatrix}8\\3\\-4\end{bmatrix}$.

11. In $\mathbb{R}^4$ let $\mathcal{P}_1$ be the hyperplane with equation $x_1 + x_2 + x_3 + x_4 = 1$, $\mathcal{P}_2$ be the hyperplane with equation $x_2 - x_3 + x_4 = 2$, and $\mathcal{P}_3$ be the hyperplane with equation $x_2 - x_4 = 3$. Find the parametric-vector equation for

    a. The intersection of $\mathcal{P}_1$ and $\mathcal{P}_2$.

b. The intersection of $\mathcal{P}_2$ and $\mathcal{P}_3$.

c. The intersection of $\mathcal{P}_1$, $\mathcal{P}_2$ and $\mathcal{P}_3$.

12. Let $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$, and $\mathbf{w} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$. Let $V = \text{Span}\,\{\mathbf{u}, \mathbf{v}\}$.

    a. Find a normal equation for the plane $V + \mathbf{w}$.

    b. Find a parametric vector equation for $V + \mathbf{w}$.

13. Let $\mathbf{p}$, $\mathbf{q}$, and $\mathbf{r}$ be vectors in $\mathbb{R}^n$. Show that the plane which contains $\mathbf{p}$, $\mathbf{q}$ and $\mathbf{r}$ can be represented by the equation $\mathbf{x} = (1 - s - t)\mathbf{p} + s\mathbf{q} + t\mathbf{r}$.

14. Suppose you are told that the equations $2x + 3y - 4z = 8$ and $\mathbf{x} = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$ represent the same plane. Which equation would be easiest to use to answer each of the following problems?

    a. Does $\begin{bmatrix} 23 \\ -6 \\ 5 \end{bmatrix}$ lie in this plane?

    b. Give 5 points which lie in this plane.

15. Suppose $V + \mathbf{v}$ and $W + \mathbf{w}$ represent the same plane then which of the following are true

    a. $V = W$. That is, V and W must be the same plane.

    b. $\mathbf{v} = \mathbf{w}$.

    c. If $\mathbf{v} \neq \mathbf{0}$ then $V + \mathbf{v}$ does not contain the origin.

    d. $V + \mathbf{v} + \mathbf{w}$ must also be the same plane as $V + \mathbf{v}$ and $W + \mathbf{w}$.

5.5 Projections

Suppose $\mathbf{u}$ and $\mathbf{v}$ are vectors in $\mathbb{R}^n$ with $\mathbf{v} \neq \mathbf{0}$. If we drop a perpendicular line from $\mathbf{u}$ onto the line determined by $\mathbf{v}$ as shown in Figure 5.8 we obtain a vector called the **projection** of $\mathbf{u}$ onto $\mathbf{v}$ which we will represent by $\text{Proj}_{\mathbf{v}}\,\mathbf{u}$.

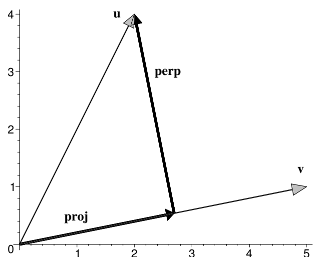


Figure 5.8. *The projection of* $\mathbf{u}$ *onto* $\mathbf{v}$.

The projection of $\mathbf{u}$ onto $\mathbf{v}$ must be in the same direction as $\mathbf{v}$ so we have $\text{Proj}_{\mathbf{v}}\mathbf{u} = k\mathbf{v}$ for some scalar k. We also know that $\mathbf{u} - \text{Proj}_{\mathbf{v}}\mathbf{u}$ is perpendicular to $\mathbf{v}$ so we can write

$$\begin{aligned}(\mathbf{u} - \text{Proj}_{\mathbf{v}}\mathbf{u}) \cdot \mathbf{v} &= 0\\ (\mathbf{u} - k\mathbf{v}) \cdot \mathbf{v} &= 0\\ \mathbf{u} \cdot \mathbf{v} - k\mathbf{v} \cdot \mathbf{v} &= 0\end{aligned}$$

We then have $k\mathbf{v} \cdot \mathbf{v} = \mathbf{u} \cdot \mathbf{v}$ and so $k = \dfrac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}}$ and

$$\text{Proj}_{\mathbf{v}}\mathbf{u} = \frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}}\mathbf{v}$$

As mentioned above the vector $\mathbf{u} - \text{Proj}_{\mathbf{v}}\mathbf{u}$ is perpendicular to $\mathbf{v}$ and is called the **orthogonal component** of the projection. We will denote this orthogonal component by $\text{Perp}_{\mathbf{v}}\mathbf{u}$

EXAMPLE 5.22. *Let* $\mathbf{u} = \begin{bmatrix}1\\4\end{bmatrix}$ *and* $\mathbf{v} = \begin{bmatrix}3\\1\end{bmatrix}$ *then*

$$\text{Proj}_{\mathbf{v}}\mathbf{u} = \frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}}\mathbf{v} = \frac{7}{10}\begin{bmatrix}3\\1\end{bmatrix} = \begin{bmatrix}21/10\\7/10\end{bmatrix}$$

and

$$\text{Perp}_{\mathbf{v}}\mathbf{u} = \mathbf{u} - \text{Proj}_{\mathbf{v}}\mathbf{u} = \begin{bmatrix}1\\4\end{bmatrix} - \begin{bmatrix}21/10\\7/10\end{bmatrix} = \begin{bmatrix}-11/10\\33/10\end{bmatrix}$$

Distance from a Point to a Line in $\mathbb{R}^n$

The method we will use for finding the distance from a point to a line is illustrated by Figure 5.8. The distance will just be $\|\text{Perp}_{\mathbf{v}}\mathbf{u}\|$.

EXAMPLE 5.23. *Find the distance from* $\begin{bmatrix}3\\1\\2\end{bmatrix}$ *to the line defined by* $\mathbf{x} = \begin{bmatrix}1\\1\\3\end{bmatrix} + t\begin{bmatrix}3\\0\\1\end{bmatrix}$.

Step 1. *Translate the line and the point so that the line goes through the origin. The simplest way to do this is to subtract* $\begin{bmatrix}1\\1\\3\end{bmatrix}$ *from the line and the point. The translated line is just* $\mathbf{x} = t\begin{bmatrix}3\\0\\1\end{bmatrix}$ *and the translated point would be* $\mathbf{u} = \begin{bmatrix}3\\1\\2\end{bmatrix} - \begin{bmatrix}1\\1\\3\end{bmatrix} = \begin{bmatrix}2\\0\\-1\end{bmatrix}$.

Step 2. *Let* $\mathbf{v} = \begin{bmatrix}3\\0\\1\end{bmatrix}$ *be the direction vector of the line and find* $\text{Proj}_{\mathbf{v}}\mathbf{u}$. *This gives*

$$\text{Proj}_{\mathbf{v}}\mathbf{u} = \frac{5}{10}\begin{bmatrix}3\\0\\1\end{bmatrix} = \begin{bmatrix}3/2\\0\\1/2\end{bmatrix}$$

Step 3. *The distance is just the length of* $\text{Perp}_{\mathbf{v}}\mathbf{u}$. *We have*

$$\text{Perp}_{\mathbf{v}}\mathbf{u} = \begin{bmatrix}2\\0\\-1\end{bmatrix} - \begin{bmatrix}3/2\\0\\1/2\end{bmatrix} = \begin{bmatrix}1/2\\0\\-3/2\end{bmatrix}$$

Thus the distance is

$$\sqrt{(1/2)^2 + (-3/2)^2} = \sqrt{5/2} = \frac{\sqrt{10}}{2}$$

Distance from a Point to a Plane

The distance from a point to a plane (or hyperplane) can be found as illustrated in Figure 5.9. The idea is that we project a vector onto a normal to the plane and the distance we want is just the length of this projection.

EXAMPLE 5.24. *Find the distance from* $\mathbf{v} = \begin{bmatrix}3\\5\\2\end{bmatrix}$ *to the plane* $x_1 + 2x_2 + 2x_3 = 6$.

We will need any point that lies on the given plane to use as a reference point. We can use the x_3 *intercept of the plane,* $\mathbf{x}_0 = \begin{bmatrix}0\\0\\3\end{bmatrix}$. *We will also need a normal to the plane,* $\mathbf{n} = \begin{bmatrix}1\\2\\2\end{bmatrix}$.

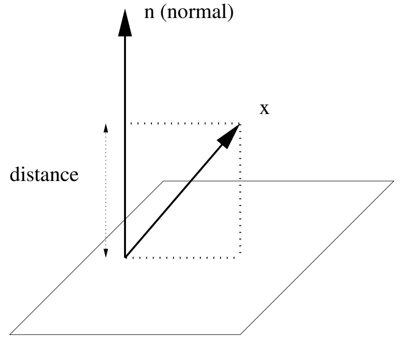


Figure 5.9. *Distance to a Plane.*

Step 1. *Translate the point and the plane so that the plane passes through the origin. We can do this by subtracting $\mathbf{x}_0$ from each. The normal to the plane would not change. The point $\mathbf{v}$ would be be translated to $\mathbf{v}-\mathbf{x}_0 = \begin{bmatrix} 3 \\ 5 \\ -1 \end{bmatrix}$.*

Step 2. *Project $\mathbf{v}-\mathbf{x}_0$ onto $\mathbf{n}$.*

Step 3. *The distance we want is just the length of this projection. In this example a few simple computations will give a distance of $11/3$.*

There is a simple formula for finding the distance from a point to a hyperplane. Suppose we have the hyperplane $\mathbf{n}^T\mathbf{x} = \mathbf{n}^T\mathbf{u}$ and the point $\mathbf{x}_0$. The point $\mathbf{u}$ lies on the hyperplane and so we can translate the hyperplane and point by subtracting $\mathbf{u}$. The hyperplane then becomes $\mathbf{n}^T\mathbf{x} = \mathbf{0}$ and the point becomes $\mathbf{x}_0 - \mathbf{u}$. Next we project the point onto the normal to the hyperplane. This projection would be

$$\frac{\mathbf{n}^T(\mathbf{x}_0 - \mathbf{u})}{\mathbf{n}^T\mathbf{n}}\mathbf{n}$$

The distance is just the length of this projection and this length is

$$\begin{aligned} \left|\frac{\mathbf{n}^T(\mathbf{x}_0 - \mathbf{u})}{\mathbf{n}^T\mathbf{n}}\right| \|\mathbf{n}\| &= \frac{\left|\mathbf{n}^T(\mathbf{x}_0 - \mathbf{u})\right|}{\|\mathbf{n}\|^2}\|\mathbf{n}\| \\ &= \frac{\left|\mathbf{n}^T\mathbf{x}_0 - \mathbf{n}^T\mathbf{u}\right|}{\|\mathbf{n}\|} \end{aligned}$$

Now if we apply the above to a hyperplane in $\mathbb{R}^2$ (that is a line) with equation $ax + by = c$ then $\mathbf{n} = \begin{bmatrix} a \\ b \end{bmatrix}$ and $\mathbf{n}^T\mathbf{u} = c$. If we let $\mathbf{x}_0 = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$ then the above formula gives us the distance

$$\frac{|ax_0 + by_0 - c|}{\sqrt{a^2+b^2}}$$

If we apply the formula to a hyperplane in $\mathbb{R}^3$ with equation $ax + by + cz = d$ then $\mathbf{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ and $\mathbf{n}^T\mathbf{u} = d$. If we let $\mathbf{x}_0 = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix}$ then the above formula gives us the distance

$$\frac{|ax_0 + by_0 + cz_0 - d|}{\sqrt{a^2+b^2+c^2}}$$

EXAMPLE 5.25. *Find the point on the plane $x + 2y + 3z = 4$ closest to $\mathbf{x}_0 = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$.*

It is easy to find the distance from $\mathbf{x}_0$ to the plane. The formula derived above tells us that this distance will be

$$\frac{|1 + 2(1) + 3(3) - 4|}{\sqrt{1+4+9}} = \frac{8}{\sqrt{14}}$$

but to find the point on the plane that lies this distance from $\mathbf{x}_0$ requires that we go back to basics. A normal to the plane is given by $\mathbf{n} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. The x intercept of the plane is $\begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix}$. If we treat this intercept as the origin the vector $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} - \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \\ 3 \end{bmatrix}$ is a vector from the plane to $\mathbf{x}_0$. Then

$$Proj_{\mathbf{n}}\mathbf{v} = \frac{1(-3) + 2(1) + 3(3)}{1+4+9}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \frac{8}{14}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 4/7 \\ 8/7 \\ 12/7 \end{bmatrix}$$

It then follows that the point closest to $\mathbf{x}_0$ is

$$\mathbf{x}_0 - Proj_{\mathbf{n}}\mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} - \begin{bmatrix} 4/7 \\ 8/7 \\ 12/7 \end{bmatrix} = \begin{bmatrix} 3/7 \\ -1/7 \\ 9/7 \end{bmatrix}$$

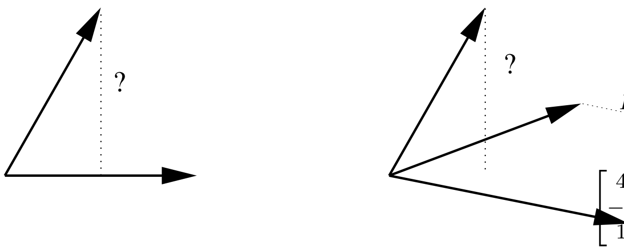
We will check this answer. Does $\begin{bmatrix} 3/7 \\ -1/7 \\ 9/7 \end{bmatrix}$ lie on the plane? Substitution into the left hand side of the equation of the plane gives

$$3/7 + 2(-1/7) + 3(9/7) = 3/7 - 2/7 + 27/7 = 28/7 = 4$$

Further computation will confirm that the distance from $\begin{bmatrix} 3/7 \\ -1/7 \\ 9/7 \end{bmatrix}$ *to* $\mathbf{x}_0$ *is* $8/\sqrt{14}$.

A General Method for Finding Distances

Every problem that involves finding a distance can be reduced to a situation as depicted in Figure 5.10.



(a) (b)

Figure 5.10. *Distance (a) to a line (b) to a plane.*

In Figure 5.10 (a) the distance you are looking for is the distance from one vector to the line generated by another vector. In Figure 5.10 (b) the distance is the distance from one vector to the plane generated by two other vectors. You should be able to see how this pattern could extend to higher dimensional situations with hyperplanes.

Such problems can always be seen as finding the distance to a subspace from some vector not in the subspace. So now suppose we want to find the distance from $\mathbf{u}$ to the subspace spanned by $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ (where these vectors are assumed to be linearly independent). We will illustrate a method for obtaining this information (and more) that seems almost magical. If you want an explanation of why this method works you will have to take a more advanced course in linear algebra. Here's the method:

- Let $A = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_k & \mathbf{u} \end{bmatrix}$.
- Compute $A^T A$ (this is sometimes called the Gram matrix).
- Put the Gram matrix $A^T A$ in row echelon form using only addrow operations (no row swaps or multiplying rows by constants). This amounts to finding matrix U of the LU decomposition that we saw earlier.
- The square root of the entry in the lower right corner is the distance you are looking for!!!

EXAMPLE 5.26. *Find the distance from* $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 2 \\ 3 \end{bmatrix}$ *to the plane generated by* $\mathbf{u}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$ *and* $\mathbf{u}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$.

Let $A = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ 0 & 1 & 2 \\ -1 & 1 & 2 \\ 1 & 1 & 3 \end{bmatrix}$. *Then* $A^T A = \begin{bmatrix} 4 & -1 & 1 \\ -1 & 4 & 8 \\ 1 & 8 & 19 \end{bmatrix}$. *Now row reduction gives*

$$\begin{bmatrix} 4 & -1 & 1 \\ -1 & 4 & 8 \\ 1 & 8 & 19 \end{bmatrix} \longrightarrow \begin{bmatrix} 4 & -1 & 1 \\ 0 & 15/4 & 33/4 \\ 0 & 33/4 & 75/4 \end{bmatrix} \longrightarrow \begin{bmatrix} 4 & -1 & 1 \\ 0 & 15/4 & 33/4 \\ 0 & 0 & 3/5 \end{bmatrix}$$

So the distance from $\mathbf{v}$ *to the span of* $\mathbf{u}_1$ *and* $\mathbf{u}_2$ *is* $\sqrt{3/5}$.

But there is more information given by this reduced matrix. The square root of the first diagonal entry is the distance of $\mathbf{u}_1$ *from the origin. The square root of the second diagonal entry is the distance from* $\mathbf{u}_2$ *to the line generated by* $\mathbf{u}_1$. *The square root of the third diagonal entry is the distance of* $\mathbf{v}$ *from the plane generated by* $\mathbf{u}_1$ *and* $\mathbf{u}_2$. *Distance, distance, distance.*

Also the product of the three diagonal entries is the square of the volume of the parallelepiped formed by the columns of A. *The product of the first two diagonal entries is the square of the area of the parallelogram formed by* $\mathbf{u}_1$ *and* $\mathbf{u}_2$. *The first diagonal entry is is the square of the length of* $\mathbf{u}_1$. *Length, area, volume!!!*

EXAMPLE 5.27. *Given the points* $A(1,0,0,1)$, $B(3,1,1,-1)$, $C(0,2,0,2)$ *in four dimensions find the distance from* C *to the line through* A *and* B.

The first step is to restate the problem in terms of vectors in $\mathbb{R}^4$. *It should be clear that the problem is equivalent to finding the distance from* $\overrightarrow{AC}$ *to the line generated by* $\overrightarrow{AB}$. *Now* $\overrightarrow{AB} = \begin{bmatrix} 2 \\ 1 \\ 1 \\ -2 \end{bmatrix}$ *and* $\overrightarrow{AC} = \begin{bmatrix} -1 \\ 2 \\ 0 \\ 1 \end{bmatrix}$.

So let $A = \begin{bmatrix} 2 & -1 \\ 1 & 2 \\ 1 & 0 \\ -2 & 1 \end{bmatrix}$ *and then* $A^T A = \begin{bmatrix} 10 & -2 \\ -2 & 6 \end{bmatrix}$. *One elementary row operation is all that's needed to put* $A^T A$ *in row echelon form giving* $\begin{bmatrix} 10 & -2 \\ 0 & 28/5 \end{bmatrix}$ *so the distance we are looking for is* $\sqrt{28/5}$. *We can also see that* $10(28/5) = 56$ *so the area of the parallelogram formed by* $\overrightarrow{AB}$ *and* $\overrightarrow{AC}$ *is* $\sqrt{56}$.

EXAMPLE 5.28. *This example is trickier and it is left up to you to figure out why it works. Find the distance from the line* $\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$ *to the plane* $\mathbf{x} = r \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} + s \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$.

Let $A = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix}$. *So* $A^T A = \begin{bmatrix} 3 & 1 & 2 & 1 \\ 1 & 2 & 1 & 0 \\ 2 & 1 & 3 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}$ *which reduces to* $\begin{bmatrix} 3 & 1 & 2 & 1 \\ 0 & 5/3 & 1/3 & -1/3 \\ 0 & 0 & 8/5 & 2/5 \\ 0 & 0 & 0 & 1/2 \end{bmatrix}$. *So the distance we want is* $\sqrt{1/2}$.

There is actually a simple explanantion why this trick works but it a bit beyond the level of an introductory course. On the other hand if we look at the specific problem of finding the distance from a point to a line we can see why it works. Suppose we want to find the distance from $\mathbf{v}$ to the line generated by $\mathbf{u}$. Then we would have $A = \begin{bmatrix} \mathbf{u} & \mathbf{v} \end{bmatrix}$. The Gram matrix would be

$$A^T A = \begin{bmatrix} \mathbf{u}^T \\ \mathbf{v}^T \end{bmatrix} \begin{bmatrix} \mathbf{u} & \mathbf{v} \end{bmatrix} = \begin{bmatrix} \mathbf{u} \cdot \mathbf{u} & \mathbf{u} \cdot \mathbf{v} \\ \mathbf{u} \cdot \mathbf{v} & \mathbf{v} \cdot \mathbf{v} \end{bmatrix}$$

One row operation reduces this to

$$\begin{bmatrix} \mathbf{u} \cdot \mathbf{u} & \mathbf{u} \cdot \mathbf{v} \\ 0 & \mathbf{v} \cdot \mathbf{v} - \frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{u} \cdot \mathbf{u}} \mathbf{u} \cdot \mathbf{v} \end{bmatrix}$$

The entry in the bottom right can be written as

$$\|\mathbf{v}\|^2 - \frac{(\mathbf{u} \cdot \mathbf{v})^2}{\|\mathbf{u}\|^2} = \|\mathbf{v}\|^2 - \|\mathbf{v}\|^2 \cos^2 \theta$$

If you've followed the reasoning so far you should be able to finish it yourself.

Notice also the fact that the determinant of the Gram matrix gives the square of the volume implies the Cauchy-Scwarz theorem that was mentioned earlier in this chapter.

Problems

1. Find $\text{Proj}_{\mathbf{v}}\mathbf{u}$ and $\text{Perp}_{\mathbf{v}}\mathbf{u}$ for

   a. $\mathbf{u} = \begin{bmatrix} 1 \\ -12 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$

   b. $\mathbf{u} = \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}$ $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$

   c. $\mathbf{u} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$

   d. $\mathbf{u} = 3\mathbf{i} + 2\mathbf{j} - \mathbf{k}$, $\mathbf{v} = \mathbf{j} - 5\mathbf{k}$

2. Find the distance from $\mathbf{x}_0$ to the given line

   a. $\mathbf{x}_0 = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$, $\mathbf{x} = t\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

   b. $\mathbf{x}_0 = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} + t\begin{bmatrix} 3 \\ -2 \end{bmatrix}$

   c. $\mathbf{x}_0 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} + t\begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}$

   d. $\mathbf{x}_0 = \begin{bmatrix} 0 \\ 2 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} 2 \\ 1 \\ 1 \\ 2 \end{bmatrix} + t\begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}$

3. Find the distance from $\mathbf{x}_0$ to the given hyperplane

   a. $\mathbf{x}_0 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$, $x - 4y = 1$

   b. $\mathbf{x}_0 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$, $3x - y - z = 4$

   c. $\mathbf{x}_0 = \begin{bmatrix} 1 \\ 3 \\ -4 \end{bmatrix}$, $x + y + 2z = 9$

   d. $\mathbf{x}_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$, $x_1 + x_2 + x_3 + x_4 = 4$

4. Find the point on the given line that is closest to $\mathbf{x}_0$

   a. $\mathbf{x}_0 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

   b. $\mathbf{x}_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + t\begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix}$

   c. $\mathbf{x}_0 = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} + t\begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$

5. Find the point on the given hyperplane that is closest to $\mathbf{x}_0$

   a. $3x + y - z = 2$m $\mathbf{x}_0 = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$

   b. $x+y-z = 4$, $\mathbf{x}_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

   c. $x_1 + 2x_2 + x_3 - x_4 = 5$, $\mathbf{x}_0 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$

6. Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{v}_4 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$.

   Find the distance

   a. from $\mathbf{v}_2$ to the span of $\mathbf{v}_1$.

   b. from $\mathbf{v}_3$ to the span of $\mathbf{v}_1$ and $\mathbf{v}_2$.

   c. from $\mathbf{v}_4$ to the span of $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$.

5.6 The Cross Product

The cross product is defined only for vectors in $\mathbb{R}^3$. Given vectors $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ then the cross product of these vectors, written $\mathbf{u} \times \mathbf{v}$, is defined by

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

Expanding this determinant by cofactors along the first row we get

$$\mathbf{u} \times \mathbf{v} = (u_2v_3 - u_3v_2)\mathbf{i} - (u_1v_3 - u_3v_1)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}$$

First of all notice that the cross product of two vectors in $\mathbb{R}^3$ is another vector in $\mathbb{R}^3$. There are a few properties of the cross product that follow directly from this definition. Try to identify the property of determinants that result in the following.

- $\mathbf{u} \times \mathbf{u} = \mathbf{0}$ for any $\mathbf{u} \in \mathbb{R}^3$.
- $\mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u}$.
- $\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) = \mathbf{v} \cdot (\mathbf{u} \times \mathbf{v}) = 0$.
- $\mathbf{u} \times (k\mathbf{v}) = k\mathbf{u} \times \mathbf{v}$
- $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}$

The first property implies that if two vectors in $\mathbb{R}^3$ are parallel then their cross product is $\mathbf{0}$. The second property says that changing the order of the factors reverses the sense of the cross product. The third property says that the cross product of two vectors is always orthogonal to both of the original factors. The **right hand rule** says that $\mathbf{u}\times\mathbf{v}$ points in the direction of your right hand thumb if you point the fingers of your right hand along $\mathbf{u}$ so that the fingers curl in the direction of $\mathbf{v}$.

EXAMPLE 5.29. *Let*

$$\mathbf{u} = \begin{bmatrix} 3 \\ 3 \\ 1 \end{bmatrix} \quad \mathbf{v} = \begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix}$$

then

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 3 & 1 \\ 2 & 5 & -1 \end{vmatrix} = -8\mathbf{i} + 5\mathbf{j} + 9\mathbf{k}$$

Notice that this cross product is orthogonal to each of the original factors as the following dot products show

$$\begin{bmatrix} 3 \\ 3 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -8 \\ 5 \\ 9 \end{bmatrix} = -24 + 15 + 9 = 0$$

$$\begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} -8 \\ 5 \\ 9 \end{bmatrix} = -16 + 25 - 9 = 0$$

Another important property of the cross product is

$$\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\| \, \|\mathbf{v}\| \sin\theta$$

where θ is the angle between $\mathbf{u}$ and $\mathbf{v}$. The proof of this is left as an exercise. This formula means that the magnitude of the cross product of two vectors gives the area of the parallelogram determined by the two vectors. This is illustrated in Figure 5.11

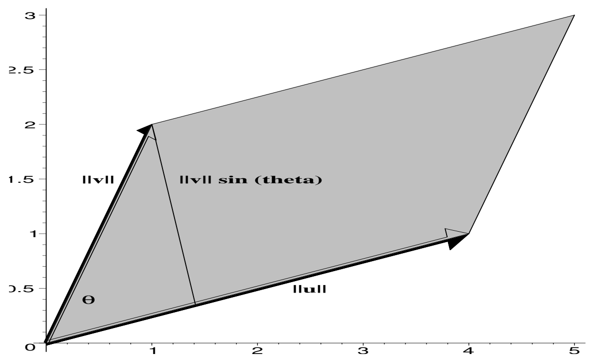


Figure 5.11.

EXAMPLE 5.30. *Given points $A(1, 2, 4)$, $B(5, 5, 4)$ and $C(3, 5, 7)$ find the area of triangle ABC.*

As usual, look at the sides of the triangle as vectors. The vectors $\overrightarrow{AB} = \begin{bmatrix} 4 \\ 3 \\ 0 \end{bmatrix}$ and $\overrightarrow{AC} = \begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix}$ determine a parallelogram. The area of this parallelogram is

$$\|\overrightarrow{AB} \times \overrightarrow{AC}\| = \left\| \begin{bmatrix} 9 \\ -12 \\ 6 \end{bmatrix} \right\| = 3\sqrt{29}$$

Triangle ABC is half of this parallelogram so the area of triangle ABC is $\frac{3\sqrt{29}}{2}$.

EXAMPLE 5.31. *We know that a plane in $\mathbb{R}^3$ can be represented by an equation of the form $\mathbf{x} = \mathbf{x_0} + s\mathbf{u} + t\mathbf{v}$ where $\mathbf{u}$ and $\mathbf{v}$ are linearly independent. Suppose we take the dot product of both sides of this equation with $\mathbf{u} \times \mathbf{v}$. We get*

$$\begin{aligned} \mathbf{x} \cdot (\mathbf{u} \times \mathbf{v}) &= (\mathbf{x_0} + s\mathbf{u} + t\mathbf{v}) \cdot (\mathbf{u} \times \mathbf{v}) \\ &= \mathbf{x_0} \cdot (\mathbf{u} \times \mathbf{v}) + s\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) + t\mathbf{v} \cdot (\mathbf{u} \times \mathbf{v}) \\ &= \mathbf{x_0} \cdot (\mathbf{u} \times \mathbf{v}) + 0 + 0 \end{aligned}$$

Let $\mathbf{n} = \mathbf{u}\times\mathbf{v}$, this will be a normal vector to the plane. The above computations can then be written $\mathbf{x}\cdot\mathbf{n} = \mathbf{x_0}\cdot\mathbf{n}$ or $\mathbf{n}^T\mathbf{x} = \mathbf{n}^T\mathbf{x_0}$, another familiar equation of the plane.

There is another way of looking at the cross product. Let

$$\mathbf{u} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \quad A = \begin{bmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{bmatrix} \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

then

$$A\mathbf{x} = \begin{bmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -cx_2 + bx_3 \\ cx_1 - ax_3 \\ -bx_1 + ax_2 \end{bmatrix} = \mathbf{u}\times\mathbf{x}$$

So a cross product can be seen as a linear transformation.

The scalar triple product

Three linearly independent vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ in $\mathbb{R}^3$ determine a parallelepiped as shown in Figure 5.12.

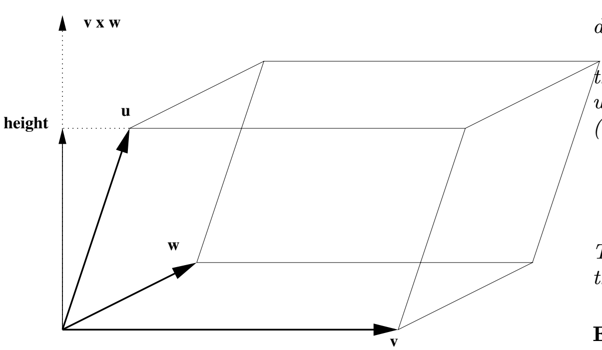


Figure 5.12. *The parallelepiped determined by $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$.*

We want to find a formula for the volume of this parallelepiped. Basic geometry tells us that the volume is the area of the base times the height. Suppose the base is determined by vectors $\mathbf{v}$ and $\mathbf{w}$. Then the area of the base is $\|\mathbf{v}\times\mathbf{w}\|$. To find the height of the parallelepiped we can project the third vector $\mathbf{u}$ onto a normal to the base. But $\mathbf{v}\times\mathbf{w}$ is normal to the base so the height is the length of

$$\text{Proj}_{\mathbf{v}\times\mathbf{w}}\mathbf{u} = \frac{\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})}{\|\mathbf{v}\times\mathbf{w}\|^2}\mathbf{v}\times\mathbf{w}$$

This means that the height is

$$\frac{|\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})|}{\|\mathbf{v}\times\mathbf{w}\|}$$

The volume of the parallelpiped is therefore given by

$$\begin{aligned} \text{Volume} &= (\text{area of base}) \quad \text{times} \quad (\text{height}) \\ &= \|\mathbf{v}\times\mathbf{w}\|\frac{|\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})|}{\|\mathbf{v}\times\mathbf{w}\|} \\ &= |\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})| \end{aligned}$$

The value $\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})$ is called the scalar triple product of $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$. It's a bit surprising but the scalar triple product can be computed as a 3×3 determinant.

$$\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \cdot \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

This means that the absolute value of a 3×3 determinant gives the volume of the parallepiped determined by the rows of the determinant (or the columns of the determinant since a matrix and its transpose have the same determinant).

EXAMPLE 5.32. *Find the volume of the parallelepiped determined by* $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$, $\mathbf{w} = \begin{bmatrix} 2 \\ 2 \\ 5 \end{bmatrix}$. *From the previous comments the volume can be easily computed using a 3×3 determinant with the given vectors as rows (or columns).*

$$\begin{vmatrix} 1 & 2 & 1 \\ 3 & 0 & 1 \\ 2 & 2 & 5 \end{vmatrix} = 0 + 4 + 6 - 0 - 2 - 30 = -22$$

The volume is the absolute value of the determinant so the volume is 22.

EXAMPLE 5.33. *Find the distance from $\mathbf{u}$ to the plane generated by $\mathbf{v}$ and $\mathbf{w}$ where*

$$\mathbf{u} = \begin{bmatrix} 3 \\ 0 \\ -2 \end{bmatrix} \quad \mathbf{v} = \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix} \quad \mathbf{w} = \begin{bmatrix} -2 \\ 1 \\ 7 \end{bmatrix}$$

There are several ways of solving this problem. The method we will use is based on the geometric insight that the distance we want is just the height of the parallelepiped with a base determined by $\mathbf{v}$ and $\mathbf{w}$ with the third side being $\mathbf{u}$.

The volume of this parallelepiped is

$$|\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})| = \begin{vmatrix} 3 & 0 & -2 \\ 4 & 1 & 1 \\ -2 & 1 & 7 \end{vmatrix} = 21 + 0 - 8 - 4 - 3 - 0 = 6$$

The area of the base is

$$\|\mathbf{v}\times\mathbf{w}\| = \left\| \begin{bmatrix} 6 \\ -30 \\ 6 \end{bmatrix} \right\| = 18\sqrt{3}$$

The height is just the volume divided by the area of the base so the distance we are looking for is

$$\frac{6}{18\sqrt{3}} = \frac{1}{3\sqrt{3}}$$

Another method would be to project $\mathbf{u}$ *onto* $\mathbf{v} \times \mathbf{w}$ *as illustrated in* **Fig 5.12**. *This gives*

$$\frac{\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})}{(\mathbf{v} \times \mathbf{w}) \cdot (\mathbf{v} \times \mathbf{w})} \mathbf{v} \times \mathbf{w} = \frac{6}{972} \begin{bmatrix} 6 \\ -30 \\ 6 \end{bmatrix}$$

The distance we want is the length of this projection which is

$$\frac{6}{972}\sqrt{972} = \frac{1}{162} 18\sqrt{3} = \frac{\sqrt{3}}{9}$$

This is equivalent to the answer found above.

Problems

1. The cross product is an example of a type of multiplication that does not satisfy the associative rule. That is, in general $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) \neq (\mathbf{u} \times \mathbf{v}) \times \mathbf{w}$. Illustrate this by evaluating $(\mathbf{i} \times \mathbf{j}) \times \mathbf{j}$ and $\mathbf{i} \times (\mathbf{j} \times \mathbf{j})$.
2. Let $\mathbf{u} = \begin{bmatrix} 2 \\ 4 \\ -1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$. Evaluate $\mathbf{u} \times \mathbf{v}$, $(\mathbf{u} + \mathbf{v}) \times (\mathbf{u} - \mathbf{v})$, and $\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v})$
3. Let $\mathbf{u} = \mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$ and $\mathbf{v} = 2\mathbf{i} - 2\mathbf{j} + \mathbf{k}$. Evaluate $\mathbf{u} \times \mathbf{v}$, $\mathbf{i} \times \mathbf{u}$, and $(\mathbf{u} + \mathbf{v}) \times (\mathbf{u} - \mathbf{v})$.
4. Find the area of the parallelogram with sides
   a. $\mathbf{u} = 2\mathbf{i}+\mathbf{j}-3\mathbf{k}$ and $\mathbf{v} = \mathbf{i} - 4\mathbf{j} + 2\mathbf{k}$
   b. $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix}$
   c. $\mathbf{u} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$
5. Use the cross product to find a normal vector to the following planes:
   a. Span $\left\{ \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} \right\}$
   b. $\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + s \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} + t \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$
   c. The plane through the origin containing $\mathbf{u} = \mathbf{i} + 2\mathbf{j}$ and $\mathbf{v} = \mathbf{i} - 2\mathbf{k}$.
   d. The plane containing the line through $\mathbf{u}$ and $\mathbf{v}$, and the line through $\mathbf{u}$ and $\mathbf{w}$ where $\mathbf{u} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$, $\mathbf{w} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$.
6. The points $A(1, 3)$, $B(3, 0)$, and $C(4, 6)$ are the vertices of a parallelogram.
   a. Find the possible values for the fourth vertex.
   b. Find the area of each possible parallelogram with these points as vertices.
7. The points $A(1, -2, 3)$, $B(3, 1, 0)$, and $C(8, 6, 4)$ are the vertices of a parallelogram.
   a. Find the possible values for the fourth vertex.
   b. Find the area of each possible parallelogram with these points as vertices.
8. Show that $(\mathbf{u} + \mathbf{v}) \times (\mathbf{u} - \mathbf{v}) = -2\mathbf{u} \times \mathbf{v}$.
9. Show that in $\mathbb{R}^2$ the area of the parallelogram determined by vectors $\mathbf{u}$ and $\mathbf{v}$ is equal to the absolute value of the 2×2 determinant having $\mathbf{u}$ and $\mathbf{v}$ as rows.
10. Find the volume of the parallelepiped with sides $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$
    a. $\mathbf{u} = 3\mathbf{i} + 2\mathbf{j}$, $\mathbf{v} = \mathbf{i} - \mathbf{j} + 3\mathbf{k}$, $\mathbf{w} = 4\mathbf{k}$.
    b. $\mathbf{u} = -\mathbf{i}+2\mathbf{j}+2\mathbf{k}$, $\mathbf{v} = 3\mathbf{i}+\mathbf{j}+2\mathbf{k}$, $\mathbf{w} = \mathbf{i}+4\mathbf{j}-\mathbf{k}$.
    c. $\mathbf{u} = \begin{bmatrix} 3 \\ 5 \\ 7 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 4 \\ 3 \\ 3 \end{bmatrix}$, $\mathbf{w} = \begin{bmatrix} 2 \\ -2 \\ -3 \end{bmatrix}$
11. Show that the parallelepiped with sides $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ has the same volume as the parallelepiped with sides $\mathbf{u}$, $\mathbf{v} - \mathbf{u}$, $\mathbf{w} - \mathbf{u}$. What can you say about the volume of the parallelepiped with sides $\mathbf{u} - \mathbf{v}$, $\mathbf{v} - \mathbf{w}$, $\mathbf{w} - \mathbf{u}$.
12. Let $\mathbf{u} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$. What geometrical objects are defined by the following sets
    a. $\{\mathbf{x} \mid \mathbf{x} \cdot \mathbf{u} = 0\}$
    b. $\{\mathbf{x} \mid \mathbf{x} \cdot \mathbf{u} = 1\}$
    c. $\{\mathbf{x} \mid \mathbf{x} \times \mathbf{u} = \mathbf{0}\}$ (note: $\mathbf{0}$ is the zero vector)
    d. $\{\mathbf{x} \mid \|\mathbf{x} \times \mathbf{u}\| = 1\}$
13. Let $A = \begin{bmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{bmatrix}$. What is the null space of A. What property of the cross product does this illustrate.
14. Suppose you are given
$$\mathbf{u} \times \mathbf{v} = \mathbf{v} \times \mathbf{w} = \mathbf{w} \times \mathbf{u}$$
    If $\mathbf{u} \times \mathbf{v} \neq \mathbf{0}$ show that $\mathbf{u} + \mathbf{v} + \mathbf{w} = \mathbf{0}$.
15. Show that
$$\begin{vmatrix} \mathbf{u} \times \mathbf{v} & \mathbf{v} \times \mathbf{w} & \mathbf{w} \times \mathbf{u} \end{vmatrix} = \begin{vmatrix} \mathbf{u} & \mathbf{v} & \mathbf{w} \end{vmatrix}^2$$
16. Let $\mathbf{u} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 1 \\ 3 \\ 3 \end{bmatrix}$. Describe the solutions to $\mathbf{x} \times \mathbf{u} = \mathbf{v}$.
17. Let $\mathbf{u}$ be a unit vector in $\mathbb{R}^3$. Define
$$T(\mathbf{x}) = (\mathbf{u} \cdot \mathbf{x})\mathbf{u} + \mathbf{u} \times \mathbf{x}$$
    a. What is $T(\mathbf{u})$?
    b. If $\mathbf{v}$ is in the subspace with normal $\mathbf{u}$ show that $T(\mathbf{v})$ is orthogonal to $\mathbf{v}$.
    c. Show that $\|T(\mathbf{x})\| = \|\mathbf{x}\|$.

5.7 Sample problems with lines and planes.

What type of intersections should you be able to find? Rather than thinking in terms of lines, or planes it is simpler to think about types of equations. There are basically three situations that can arise:

- One parametric-vector equation and one normal (or Cartesian) equation.
- Any number of normal equations. (This situation just involves solving a system of linear equations.)
- Two parametric-vector equations.

EXAMPLE 5.34. *Find the point of intersection of the following lines*

$$\mathbf{x} = \begin{bmatrix}2\\1\\0\end{bmatrix} + s\begin{bmatrix}1\\1\\3\end{bmatrix} \qquad \mathbf{x} = \begin{bmatrix}11\\2\\5\end{bmatrix} + t\begin{bmatrix}3\\-1\\-2\end{bmatrix}$$

The point of intersection would occurs when these two equations are equal so we begin by equating the right hand sides.

$$\begin{bmatrix}2\\1\\0\end{bmatrix} + s\begin{bmatrix}1\\1\\3\end{bmatrix} = \begin{bmatrix}11\\2\\5\end{bmatrix} + t\begin{bmatrix}3\\-1\\-2\end{bmatrix}$$

Rearranging these terms we get

$$s\begin{bmatrix}1\\1\\3\end{bmatrix} + t\begin{bmatrix}-3\\1\\2\end{bmatrix} = \begin{bmatrix}11\\2\\5\end{bmatrix} - \begin{bmatrix}2\\1\\0\end{bmatrix} = \begin{bmatrix}9\\1\\5\end{bmatrix}$$

We set up and reduce the corresponding augmented matrix.

$$\begin{bmatrix}1 & -3 & 9\\1 & 1 & 1\\3 & 2 & 5\end{bmatrix} \sim \begin{bmatrix}1 & 0 & 3\\0 & 1 & -2\\0 & 0 & 0\end{bmatrix}$$

This reduced form gives us the values of $s = 3$ and $t = -2$ corresponding to the point of intersection. The value $s = 3$ gives

$$\mathbf{x} = \begin{bmatrix}2\\1\\0\end{bmatrix} + 3\begin{bmatrix}1\\1\\3\end{bmatrix} = \begin{bmatrix}5\\4\\9\end{bmatrix}$$

This would be the point of intersection.

EXAMPLE 5.35. *Find the intersection of the two planes*

$$\mathbf{x} = s_1\begin{bmatrix}1\\1\\1\\1\end{bmatrix} + s_2\begin{bmatrix}1\\2\\3\\4\end{bmatrix} \quad \text{and} \quad \mathbf{x} = t_1\begin{bmatrix}1\\3\\3\\1\end{bmatrix} + t_2\begin{bmatrix}3\\1\\1\\3\end{bmatrix} + \begin{bmatrix}2\\0\\-3\\5\end{bmatrix}$$

We set these expressions equal to each othe

$$s_1\begin{bmatrix}1\\1\\1\\1\end{bmatrix} + s_2\begin{bmatrix}1\\2\\3\\4\end{bmatrix} = t_1\begin{bmatrix}1\\3\\3\\1\end{bmatrix} + t_2\begin{bmatrix}3\\1\\1\\3\end{bmatrix} + \begin{bmatrix}2\\0\\-3\\5\end{bmatrix}$$

and rearrange

$$s_1\begin{bmatrix}1\\1\\1\\1\end{bmatrix} + s_2\begin{bmatrix}1\\2\\3\\4\end{bmatrix} - t_1\begin{bmatrix}1\\3\\3\\1\end{bmatrix} - t_2\begin{bmatrix}3\\1\\1\\3\end{bmatrix} = \begin{bmatrix}2\\0\\-3\\5\end{bmatrix}$$

Set up the corresponding augmented matrix and reduce

$$\begin{bmatrix}1 & 1 & 1 & 3 & 2\\1 & 2 & 3 & 1 & 0\\1 & 3 & 3 & 1 & -3\\1 & 4 & 1 & 3 & 5\end{bmatrix} \sim \begin{bmatrix}1 & 1 & 1 & 3 & 2\\0 & 1 & 2 & -2 & -2\\0 & 0 & -2 & 2 & -5\\0 & 0 & 0 & 0 & 12\end{bmatrix}$$

The system is inconsistent. These planes don't intersect.

In $\mathbb{R}^2$ two non-parallel lines must intersect. In $\mathbb{R}^3$ this is not true, because in $\mathbb{R}^3$ there is more "room" for the lines to move around in and avoid touching. In $\mathbb{R}^3$ two non-parallel planes must intersect, but as this example shows this is not true in $\mathbb{R}^4$. This is again due to the fact that there is more "room" in $\mathbb{R}^4$ where the planes may be located.

You have to be careful when trying to extend your intuition to higher dimensional spaces. Look at the reduced matrix we got in this example. Notice that if the four generating vectors had been linearly independent these planes would have intersected in a single point! On the other hand, if we had altered the translation vector in the second plane in such a way that the reduced matrix had a row of zeroes, then the planes would intersect along a straight line.

You should understand the two basic types of equations. A parametric-vector equation can be used to represent a flat of any dimension. A Cartesian equation (i.e., a linear equation or normal equation) can only represent a hyperplane - that is, a object of dimension one less than the surrounding space.

EXAMPLE 5.36. *For what values of a and b do $\mathbf{x} = \begin{bmatrix}5\\1\end{bmatrix} + t\begin{bmatrix}2\\1\end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix}a\\3\end{bmatrix} + t\begin{bmatrix}b\\2\end{bmatrix}$ represent the same line?*

The lines must have the same directions and so

$$\begin{bmatrix}b\\2\end{bmatrix} = k\begin{bmatrix}2\\1\end{bmatrix}$$

Looking at the second component of these vectors we see that $k = 2$ and so $b = 4$.

We also know that $\begin{bmatrix}5\\1\end{bmatrix}$ *lies on the line so*

$$\begin{bmatrix}5\\1\end{bmatrix} = \begin{bmatrix}a\\3\end{bmatrix} + t\begin{bmatrix}4\\2\end{bmatrix} = \begin{bmatrix}a+4t\\3+2t\end{bmatrix}$$

The second component tells us that $1 = 3 + 2t$ *and so* $t = -1$. *Substituting this into the first component gives* $5 = a - 1(4)$ *and so* $a = 9$.

The equation we are looking for is then

$$\mathbf{x} = \begin{bmatrix}9\\3\end{bmatrix} + t\begin{bmatrix}4\\2\end{bmatrix}$$

EXAMPLE 5.37. *Find the normal equation of the plane which contains the line* $\mathbf{x} = \begin{bmatrix}1\\1\\2\end{bmatrix} + t\begin{bmatrix}2\\-1\\-1\end{bmatrix}$ *and the point* $\begin{bmatrix}4\\3\\5\end{bmatrix}$.

We will find two vectors parallel to the plane and then take their cross product. This cross product will be a normal vector to the plane.

The direction vector of the given line is $\mathbf{u} = \begin{bmatrix}2\\-1\\-1\end{bmatrix}$ *and this vector must be parallel to the plane since the plane contains the given line.*

The points $\begin{bmatrix}1\\1\\2\end{bmatrix}$ *and* $\begin{bmatrix}4\\3\\5\end{bmatrix}$ *both lie in the plane. We will use* $\begin{bmatrix}1\\1\\2\end{bmatrix}$ *as a reference point and then the vector* $\mathbf{v} = \begin{bmatrix}4\\3\\5\end{bmatrix} - \begin{bmatrix}1\\1\\2\end{bmatrix} = \begin{bmatrix}3\\2\\3\end{bmatrix}$ *must also be parallel to the plane.*

The cross product $\mathbf{v}\times\mathbf{u}$ *is a normal vector to the plane. The cross product would be*

$$\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 3 & 2 & 3\\ 2 & -1 & -1\end{vmatrix} = \mathbf{i} + 9\mathbf{j} - 7\mathbf{k}$$

The normal equation of the plane is then $x_1 + 9x_2 - 7x_3 = d$. *It remains to find* d *and this can be done by substituting one of the points that lies on the plane. If we use the point* $\begin{bmatrix}1\\1\\2\end{bmatrix}$ *we have*

$$1 + 9(1) - 7(2) = -4 = d$$

The normal equation of the plane is therefore $x_1 + 9x_2 - 7x_3 = -4$.

You should be able to find the distance between various types of flats 0-flats (points), 1-flats (lines), 2-flates (planes), etc. There are many ways of doing this but the use of projections is the most general.

EXAMPLE 5.38. *Let* $\mathcal{L}$ *be the line* $\mathbf{x} = t\begin{bmatrix}1\\2\\2\end{bmatrix}$ *and let* $\mathbf{u} = \begin{bmatrix}3\\1\\3\end{bmatrix}$. *Find the distance from* $\mathbf{u}$ *to* $\mathcal{L}$ *and find the point on* $\mathcal{L}$ *that is closest to* $\mathbf{u}$. *Finally, find the parametric-vector equation of the line through* $\mathbf{u}$ *perpendicular to* $\mathcal{L}$.

Let $\mathbf{v} = \begin{bmatrix}1\\2\\2\end{bmatrix}$.

The projection of $\mathbf{u}$ *onto* $\mathcal{L}$ *would be*

$$\mathrm{Proj}_{\mathcal{L}}\mathbf{u} = \frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\mathbf{v} = \frac{11}{9}\begin{bmatrix}1\\2\\2\end{bmatrix} = \begin{bmatrix}11/9\\22/9\\22/9\end{bmatrix}$$

The orthogonal component of the projection is then

$$\mathrm{Perp}_{\mathcal{L}}\mathbf{u} = \mathbf{u} - \mathrm{Proj}_{\mathcal{L}}\mathbf{u} = \begin{bmatrix}3\\1\\3\end{bmatrix} - \begin{bmatrix}11/9\\22/9\\22/9\end{bmatrix} = \begin{bmatrix}16/9\\-13/9\\5/9\end{bmatrix}$$

The distance from $\mathbf{u}$ *to* $\mathcal{L}$ *is the length of the orthogonal component,* $1/9\sqrt{16^2 + (-13)^2 + 5^2} = 1/9\sqrt{450} = 5\sqrt{2}/3$

The point on $\mathcal{L}$ *closest to* $\mathbf{u}$ *is just the projection found above,* $\begin{bmatrix}11/9\\22/9\\22/9\end{bmatrix}$

The equation of the line perpendicular to $\mathcal{L}$ *passing through* $\mathbf{u}$ *is now easy.*

$$\mathbf{x} = \begin{bmatrix}3\\1\\3\end{bmatrix} + t\begin{bmatrix}16/9\\-13/9\\5/9\end{bmatrix}$$

EXAMPLE 5.39. *Find the distance from* $\mathbf{u} = \begin{bmatrix}3\\0\\2\end{bmatrix}$ *to the plane* $x_1 + x_2 + 2x_3 = 10$.

Find a point on the plane. For example, the x_1 *intercept* $\mathbf{v} = \begin{bmatrix}10\\0\\0\end{bmatrix}$. *This point will be used as our reference point or origin.*

Let $\mathbf{w} = \mathbf{u} - \mathbf{v} = \begin{bmatrix}-7\\0\\2\end{bmatrix}$.

The situation is now the same as that illustrated in **Figure 5.9**. *The distance we are looking for is just the length of the projection of* $\mathbf{w}$ *onto the normal vector of the plane,* $\mathbf{n} = \begin{bmatrix}1\\1\\2\end{bmatrix}$.

$$\text{Proj}_{\mathbf{n}}\mathbf{w} = \frac{\mathbf{w}\cdot\mathbf{n}}{\mathbf{n}\cdot\mathbf{n}}\mathbf{n} = \frac{-3}{6}\begin{bmatrix}1\\1\\2\end{bmatrix}$$

The length of this projection is $\frac{3}{6}\sqrt{6} = \sqrt{6}/2$.

EXAMPLE 5.40. *Find the distance between the two skew lines*

$$\mathcal{L}_1: \quad \mathbf{x} = \begin{bmatrix}1\\1\\0\end{bmatrix} + s\begin{bmatrix}2\\3\\1\end{bmatrix} \quad \text{and} \quad \mathcal{L}_2: \quad \mathbf{x} = \begin{bmatrix}3\\1\\3\end{bmatrix} + t\begin{bmatrix}0\\2\\-1\end{bmatrix}$$

This type of problem is more difficult to visualize. We are trying to find the shortest distance between two non-intersecting lines in $\mathbb{R}^3$. *Imagine translating* $\mathcal{L}_2$ *so that it intersects with* $\mathcal{L}_1$. *Then* $\mathcal{L}_1$ *and the translated* $\mathcal{L}_2$ *will lie in a plane and* $\mathcal{L}_2$ *is parallel to this plane. The distance we are looking for is the distance from* $\mathcal{L}_2$ *to that plane. So the problem reduces down to finding the distance from a point (any point on* $\mathcal{L}_2$*) to a plane.*

The vectors $\begin{bmatrix}2\\3\\1\end{bmatrix}$ *and* $\begin{bmatrix}0\\2\\-1\end{bmatrix}$ *are parallel to the plane so their cross product would be normal to the plane.*

$$\begin{bmatrix}2\\3\\1\end{bmatrix} \times \begin{bmatrix}0\\2\\-1\end{bmatrix} = \begin{bmatrix}-5\\2\\4\end{bmatrix}$$

Now $\begin{bmatrix}1\\1\\0\end{bmatrix}$ *lies in the plane and* $\begin{bmatrix}3\\1\\3\end{bmatrix}$ *lies on* $\mathcal{L}_2$ *and so*

$$\begin{bmatrix}3\\1\\3\end{bmatrix} - \begin{bmatrix}1\\1\\0\end{bmatrix} = \begin{bmatrix}2\\0\\3\end{bmatrix}$$

must be projected onto the normal. This gives

$$\frac{\begin{bmatrix}2\\0\\3\end{bmatrix}\cdot\begin{bmatrix}-5\\2\\4\end{bmatrix}}{\begin{bmatrix}-5\\2\\4\end{bmatrix}\cdot\begin{bmatrix}-5\\2\\4\end{bmatrix}}\begin{bmatrix}-5\\2\\4\end{bmatrix} = \frac{2}{45}\begin{bmatrix}-5\\2\\4\end{bmatrix}$$

The length of this projection is $\dfrac{2}{45}\sqrt{45}$ *and this is the distance we are looking for.*

Problems

1. Given the points $A(1,3,3)$, $B(0,2,-1)$, and $C(3,2,2)$ find
   a. The area of triangle ABC.
   b. A normal equation of the plane containing A, B and C.
   c. A parametric vector equation of the plane containing A, B an C.
   d. An equation of the line through A and B.
2. Given the points $A(0,2,1)$, $B(1,1,-1)$, and $C(5,3,5)$ find
   a. The area of triangle ABC.
   b. A normal equation of the plane containing A, B and C.
   c. a parametric vector equation of the plane containing A, B and C.
   d. An equation of the line through A and B.
3. Let $\mathcal{L}_1$ be the line $\mathbf{x} = s\begin{bmatrix}1\\2\\1\end{bmatrix}$ and $\mathcal{L}_2$ be the line $\mathbf{x} = t\begin{bmatrix}1\\0\\-1\end{bmatrix}$. Let $\mathcal{P}$ be the plane containing $\mathcal{L}_1$ and $\mathcal{L}_2$. Let $\mathbf{v} = \begin{bmatrix}2\\-1\\3\end{bmatrix}$.
   a. Find the distance from $\mathbf{v}$ to $\mathcal{L}_1$.
   b. Find the distance from $\mathbf{v}$ to $\mathcal{L}_2$.
   c. Find the distance from $\mathbf{v}$ to $\mathcal{P}$.
4. Let $\mathcal{L}_1$ be the line $\mathbf{x} = \begin{bmatrix}2\\1\\1\end{bmatrix} + s\begin{bmatrix}1\\1\\2\end{bmatrix}$ and $\mathcal{L}_2$ be the line $\mathbf{x} = \begin{bmatrix}0\\-1\\-3\end{bmatrix} + t\begin{bmatrix}1\\-1\\1\end{bmatrix}$. Let $\mathbf{v} = \begin{bmatrix}3\\2\\5\end{bmatrix}$.
   a. Show that $\mathcal{L}_1$ and $\mathcal{L}_2$ intersect.
   b. Find the distance from $\mathbf{v}$ to $\mathcal{L}_1$.
   c. Find the distance from $\mathbf{v}$ to $\mathcal{L}_2$.
   d. Find the distance from $\mathbf{v}$ to $\mathcal{P}$, the plane containing $\mathcal{L}_1$ and $\mathcal{L}_2$.
5. Find the distance between the parallel lines $x+2y = 3$ and $x+2y = 0$.
6. Find the distance between the skew lines $\mathbf{x} = \begin{bmatrix}1\\1\\0\end{bmatrix} + s\begin{bmatrix}1\\3\\2\end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix}2\\1\\1\end{bmatrix} + t\begin{bmatrix}1\\0\\3\end{bmatrix}$
7. Find the distance between the parallel hyperplanes $4x_1+2x_2+2x_3+x_4 = 2$ and $4x_1+2x_2+2x_3+x_4 = 5$.
8. The line $\mathbf{x} = \begin{bmatrix}4\\0\\-1\end{bmatrix} + t\begin{bmatrix}2\\3\\1\end{bmatrix}$ lies in which of the following planes?
   a. $2x+3y+z = 7$
   b. $x+y-5z = 9$
   c. $2x-y-z = 9$

5.8 Maple examples

EXAMPLE 5.41. *Given the lines*

$$\mathbf{x} = \begin{bmatrix}1\\3\\2\end{bmatrix} + s\begin{bmatrix}4\\1\\-1\end{bmatrix} \quad \mathbf{x} = \begin{bmatrix}8\\-4\\9\end{bmatrix} + t\begin{bmatrix}1\\2\\-2\end{bmatrix}$$

we will find the point of intersection and plot the lines along with the plane containing the lines.

*We will start be defining the relevant vectors in **Maple** .*

```
>with(LinearAlgebra):   ### you always need this
>with(plots):    ### we need this for some plots
>v1:=<1,3,2>:v2:=<4,1,-1>:
>u1:=<8,-4,9>:u2:=<1,2,-2>:
>A:=<v2 | -u2 | u1-v1>:
>ReducedRowEchelonForm(A);
```

$$\begin{bmatrix}1 & 0 & 3\\0 & 1 & 5\\0 & 0 & 0\end{bmatrix}$$

So we have $s = 3$ and $t = 5$.

```
>v1+3*v2;
          [ 13, 6, -1]
>u1+5*u2;
          [13, 6, -1]
```

The point of intersection is $\begin{bmatrix}13\\6\\-1\end{bmatrix}$.

You can now plot the lines and the plane containing the lines.

```
>L1:=v1+s*v2:
>L2:=u1+t*u2:
>P1:=s*v2+t*u2+v1:  ### Span(v2,u2) translated
>line1:=spacecurve(L1,s=0..5,color=black,thickness=2):
>line2:=spacecurve(L2,t=0..8,color=blue,thickness=2):
>plane1:=plot3d(P1,s=-3..2,t=-3..2,style=patchnogrid):
>display([line1,line2,plane1],axes=boxed);
```

You can rotate the plot using the mouse to view the image from different points of view.

EXAMPLE 5.42. *We will use **Maple** to find the distance between two skew lines in $\mathbb{R}^3$. We will then find the points on the two lines that are closest. The lines will be*

$$\mathcal{L}_1 : \quad \mathbf{x} = \begin{bmatrix}1\\4\\1\end{bmatrix} + s\begin{bmatrix}2\\4\\1\end{bmatrix}$$

and

$$\mathcal{L}_2 : \quad \mathbf{x} = \begin{bmatrix}3\\0\\3\end{bmatrix} + t\begin{bmatrix}1\\3\\-1\end{bmatrix}$$

We start by plotting the lines.

```
>L1:=<1,4,1>+s*<2,4,1>:
>L2:=<3,0,3>+t*<1,3,-1>:
>p1:=spacecurve(L1,s=-3..3,thickness=2,color=blue):
>p2:=spacecurve(L2,t=-3..3,thickness=2,color=black):
>display(p1,p2);
```

We proceed as we did in the text. The key insight is that the direction vectors for these lines, $\begin{bmatrix}2\\4\\1\end{bmatrix}$ *and* $\begin{bmatrix}1\\3\\-1\end{bmatrix}$, *define a plane parallel to both lines. A normal vector,* $\mathbf{n}$, *to this plane will be*

```
>n:=CrossProduct(<2,4,1>, <1,3,-1>);
                         n = [-7, 3, 2]
```

We will translate both lines by subtracting $\begin{bmatrix}1\\4\\1\end{bmatrix}$. *After translation $\mathcal{L}_1$ will pass through the origin. The translated $\mathcal{L}_2$ will not pass through the origin but*

$$\mathbf{v} = \begin{bmatrix}3\\0\\3\end{bmatrix} - \begin{bmatrix}1\\4\\1\end{bmatrix} = \begin{bmatrix}2\\-4\\2\end{bmatrix}$$

will lie on this line.

```
>v:=<3,0,3>-<1,4,1>;
                  v = [2, -4, 2]
```

The distance we are looking for will then be the distance from $\mathbf{v}$ to the plane through the origin with normal $\mathbf{n}$. We find this distance as usual.

```
>u:=DotProduct(v,n)/DotProduct(n,n)*n;
>Norm(u,2);
```

The last command gives the distance as displaystyle$\frac{11\sqrt{62}}{31}$.

Now the vector $\mathbf{u}$ represents the vector from $\mathcal{L}_1$ to $\mathcal{L}_2$ at the closest points. In general, the a vector from $\mathcal{L}_1$ to $\mathcal{L}_2$ would be

```
>w:=evalm(L2-L1):
              w = [2+t-2*s, 3*t-4-4*s, 2-t-s]
```

*For what values of s and t will $\mathbf{w}$ be equal to $\mathbf{u}$? We can answer this in **Maple** as follows*

```
>solve( {w[1]=u[1], w[2]=u[2], w[3]=u[3]}, {s,t});
                 s= 23/31 ,  t = 61/31
```

The two closest points are therefore

```
>x1:=<1,4,1>+23/31*<2,4,1>:
                    x1 = [77/31, 216/31, 54/31]
>x2:=<3,0,3>+61/31*<1,3,-1>:
                    x2 = [154/31, 183/31, 32/31]
```

Here is another plot which shows the lines and the closest points

```
>p3:=pointplot3d( {x1, x2}, connect=true, color=red):
>display( {p1,p2,p3});
```

An alternate way of finding the distance between the lines (as described in **Section 5.5** *is as follows*

```
>A:=<<2,4,1>|<-1,-3,1>|<2,-4,2>>:
>GaussianElimination(Transpose(A).A);
```

$$\begin{bmatrix} 21 & -13 & -10 \\ 0 & \frac{62}{21} & \frac{122}{21} \\ 0 & 0 & \frac{242}{31} \end{bmatrix}$$

The distance between the lines is therefore $\sqrt{\dfrac{242}{31}} = \dfrac{11\sqrt{2}}{\sqrt{31}}$.

EXAMPLE 5.43. *We will use* ***Maple*** *to prove the formula* $\|\mathbf{u}\times\mathbf{v}\| = \|\mathbf{u}\|\,\|\mathbf{v}\|\sin\theta$ *where* θ *is the angle between* $\mathbf{u}$ *and* $\mathbf{v}$.

We will actually show that $\|\mathbf{u}\times\mathbf{v}\|^2 = \|\mathbf{u}\|^2\,\|\mathbf{v}\|^2\sin^2\theta$ *and the given formula follows directly.*

First we define two arbitrary vectors in $\mathbb{R}^3$.

```
>u:=<u1,u2,u3>:
>v:=<v1,v2,v3>:
>uv:=CrossProduct(u,v): ### uv is the cross product of u and v
```

We will find an expression for $\sin^2\theta$ *using the formula* $\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|}$ *and* $\sin^2\theta = 1-\cos^2\theta$. *We multiply by the transpose to compute the dot product. We start with* $\cos^2\theta$

```
>c2:=(u^%T.v)^2/u^%T.u/v^%T.v:
>s2:=1-c2:
```

Next we want $\|\mathbf{u}\times\mathbf{v}\|^2$ *which we will divide by* $\sin^2\theta$.

```
>normuv2:=uv^%T.uv:
>simplify(normuv2/s2);
```

$$\left(u1^2+u2^2+u3^2\right)\left(v1^2+v2^2+v3^2\right)$$

This last result is clearly $\|\mathbf{u}\|^2\,\|\mathbf{v}\|^2$. *This shows that*

$$\frac{\|\mathbf{u}\times\mathbf{v}\|^2}{\sin^2\theta} = \|\mathbf{u}\|^2\,\|\mathbf{v}\|^2$$

and the desired formula follows easily.

EXAMPLE 5.44. *Find the distance from* $\begin{bmatrix}0\\t\\0\\1\end{bmatrix}$ *to the plane generated by* $\begin{bmatrix}1\\1\\1\\1\end{bmatrix}$ *and* $\begin{bmatrix}1\\-1\\1\\0\end{bmatrix}$.

Here's one way:

```
>A:=<<1,1,1,1>,<1,-1,1,0>,<0,t,0,1>>:
>B:=Transpose(A).A:
>U:=LUDecomposition(B,output='U');
>dist:=sqrt(U[3,3]);
```

Matrix U is the result of applying Gaussian elimination (using only addrows) to A^TA. *This gives*

$$U = \begin{bmatrix} 4 & 1 & 1+t \\ 0 & 11/4 & -5/4\,t-1/4 \\ 0 & 0 & \frac{8}{11}+2/11\,t^2-\frac{8}{11}\,t \end{bmatrix}$$

Using the information given by the diagonal entries the distance we want is the square root of the entry in the third row and third column. So the distance is $\dfrac{\sqrt{22}\sqrt{(-2+t)^2}}{11}$. *Notice that this implies that when* $t=2$ *the distance is 0. In other words when* $t=2$ *the third column of* A *lies in the plane generated by the first two columns.*

Summary

Any $n-1$ dimensional subspace, V, in $\mathbb{R}^n$ can be represented by an equation of the form

$$\mathbf{n}^T\mathbf{x} = 0$$

where $\mathbf{n}$ is a normal vector to the subspace. The subspace consists of all vectors orthogonal to $\mathbf{n}$.

This idea can be generalized in the following way: the equation

$$\mathbf{n}^T\mathbf{x} = \mathbf{n}^T\mathbf{x_0}$$

represents a hyperplane in $\mathbb{R}^n$ with normal $\mathbf{n}$ and which contains vector $\mathbf{x_0}$. This hyperplane is $V + \mathbf{x_0}$, V translated by $\mathbf{x_0}$.

A system of linear equations with n unknowns corresponds to a collection of hyperplanes in $\mathbb{R}^n$. Solving the system corresponds to finding the intersection of the hyperplanes. The solution is always some subspace of $\mathbb{R}^n$ with a possible translation.

Lines, planes, etc. in $\mathbb{R}^n$ which contain the origin are called linear subspaces of $\mathbb{R}^n$. If they are translated away from the origin they are called affine subspaces or k-flats.

- A 1-flat is a line and has a parametric-vector equation of the form $\mathbf{x} = \mathbf{x_0} + s\mathbf{u}$.
- A 2-flat is called a plane and has a parametric-vector equation of the form $\mathbf{x} = \mathbf{x_0} + s\mathbf{u} + t\mathbf{v}$.
- A 3-flat has a parametric-vector equation of the form $\mathbf{x} = \mathbf{x_0} + r\mathbf{u} + s\mathbf{v} + t\mathbf{w}$. There is no standard term (such as plane or line) to refer to this type of space.
- In general an n-flat has an equation of the form $\mathbf{x} = \mathbf{x_0} + t_1\mathbf{u}_1 + t_2\mathbf{u}_2 + \cdots + t_n\mathbf{u}_n$.

We began this course with the problem of solving a system of equations and we now have a geometric interpretation of this procedure. A system of equations can now be seen as a collection of hyperplanes. The solution to such a system is the intersection of all these hyperplanes and this intersection is a k-flat (i.e., a translated linear subspace) where k is the number of parameters needed in the general solution. When we solve a system we are just giving the equation for this k-flat in parametric-vector form.

ANSWERS

Section 5.1 1. $(7,-3)$ 2. $(11,-6,5)$ 3. (a) $\sqrt{13}$ (b) $3\sqrt{2}$ (c) $\sqrt{30}$ (d) 1 (e) 1 (f) $\sqrt{3}$ (g) $\sqrt{26}$ (h) 1 4. $\sqrt{n}$ 5. (a) $3\sqrt{2}$, $\sqrt{26}$, $\sqrt{20}$ (b) $\sqrt{13}$, $\sqrt{17}$, $\sqrt{20}$ (c) all $\sqrt{2}$ (d) $\sqrt{21}$, $\sqrt{27}$, $\sqrt{46}$ 6. (a) (2,11) (b) (-8,5,1) 7. $k=4/3$ 8. (a) $\begin{bmatrix}3/5\\4/5\end{bmatrix}$ (b) $\begin{bmatrix}3/\sqrt{50}\\4/\sqrt{50}\\5/\sqrt{50}\end{bmatrix}$ (c) $\begin{bmatrix}1/2\\1/2\\-1/2\\-1/2\end{bmatrix}$ (d) $3/\sqrt{38}\mathbf{i}-5/\sqrt{38}\mathbf{i}+2/\sqrt{38}\mathbf{k}$ (e) $\begin{bmatrix}1/\sqrt{1+t^2}\\t/\sqrt{1+t^2}\end{bmatrix}$ (f) $1/\sqrt{2}\begin{bmatrix}\cos t+\sin t\\\cos t-\sin t\end{bmatrix}$ 9. $2\le\|AC\|\le 8$ 10. Hint: you need the trig identity $\cos(A-B)=\cos A\cos B+\sin A\sin B$

12. (a) F (b) T (c) T (d) F (e) F. (f) T 13. When $\mathbf{u}$ and $\mathbf{v}$ have the same direction and sense. 14. Use $\|A\mathbf{v}\|^2=\mathbf{v}^TA^TA\mathbf{v}$

Section 5.2 1. (a)$\sqrt{5}$, 5, 2, 79.70° (b) $\sqrt{26}$, 3, -3, 101.31° (c) $\sqrt{3}$, $\sqrt{14}$, 2, 72.02° (d) 22.52° (e) $\arccos(2ab/(a^2+b^2))$ (f) 54.74° 2. (a) 40.60°, 63.43°, 75.96° (b) 75.04°, 43.09°, 61.87° (c) 90.00°, 45.00°, 45.00° 3. $k=4,0$ 4. 6. (a) $k=\pm\sqrt{6}$ (b) $k=-2/3$ (c) $k=6$ 7. (a)$k=\pm 2$ (b) $k=7$ (c) none (d) $\frac{-1\pm\sqrt{7}}{2}$ 8. 9. (a) $\arccos(1/3)$, $\arccos(2/3)$, $\arccos(-2/3)$ 10 (a) $\pi/6$ or $5\pi/6$. (b)any angle between$\pi/6$ and $2\pi/3$.

Section 5.3 1. (a)$\mathbf{x}=\begin{bmatrix}0\\5\end{bmatrix}+t\begin{bmatrix}1\\-3\end{bmatrix}$ (b)$\mathbf{x}=\begin{bmatrix}3\\1\end{bmatrix}+t\begin{bmatrix}5\\2\end{bmatrix}$ (c) $\mathbf{x}=\begin{bmatrix}4\\0\end{bmatrix}+t\begin{bmatrix}0\\1\end{bmatrix}$ (d) $\mathbf{x}=\begin{bmatrix}0\\-3\end{bmatrix}+t\begin{bmatrix}1\\0\end{bmatrix}$ 2. (a) $x_1+3x_2=5$ (b) $2x_1-5x_2=-15$ (c)$3x_1-4x_2=0$ (d) $x_2=x_1-1$ 3. (a)(-2,-1), (-5,5) (b) $t=4$ (c) $\begin{bmatrix}3\pm 2/\sqrt{5}\\-1\mp 4/\sqrt{5}\end{bmatrix}$ (d) (0,-5), (-4,3) 4.(a) (3,-1,2), (0,2,-1) (b) $t=-2/3$ (c) $\begin{bmatrix}2\pm 2/\sqrt{3}\\\mp 2/\sqrt{3}\\1\pm 2/\sqrt{3}\end{bmatrix}$ (d) 5.: $a=-1$, $b=8$ 6. $a=16/3$, $b=4/3$ 7. $a=1/2$, $b=3/2$ 8. 9. (a) $\mathbf{x}=\begin{bmatrix}1\\2\\3\end{bmatrix}+t\begin{bmatrix}3\\-1\\-2\end{bmatrix}$ (b) $\mathbf{x}=\begin{bmatrix}2\\2\\5\\5\end{bmatrix}+t\begin{bmatrix}2\\0\\4\\8\end{bmatrix}$ 10. 11. a=3,b=2 12. 13.

Section 5.4 1. (a)$x_1+2x_2-x_3=1$ (b)$3x_1+3x_2=9$ (c)$x_1+x_2+3x_3=-2$ (d)$3x_1+x_2+2x_3+2x_4=10$ 2. (a) $\mathbf{x}=\begin{bmatrix}3\\0\\0\end{bmatrix}+s\begin{bmatrix}1\\-1\\0\end{bmatrix}+t\begin{bmatrix}2\\0\\1\end{bmatrix}$ (b)$\mathbf{x}=\begin{bmatrix}0\\0\\5\end{bmatrix}+s\begin{bmatrix}1\\0\\-3\end{bmatrix}+t\begin{bmatrix}0\\1\\-2\end{bmatrix}$ (c)$\mathbf{x}=\begin{bmatrix}1\\0\\0\\0\end{bmatrix}+r\begin{bmatrix}1\\-1\\0\\0\end{bmatrix}+s\begin{bmatrix}1\\0\\-1\\0\end{bmatrix}+t\begin{bmatrix}1\\0\\0\\-1\end{bmatrix}$ 3. basis: $\begin{bmatrix}-3\\1\\0\end{bmatrix}$, $\begin{bmatrix}1\\0\\1\end{bmatrix}$ line: $\mathbf{x}=t\begin{bmatrix}1\\3\\-1\end{bmatrix}$ 4. like number 3 5. $x-2y+4z=5$ 6.$\mathbf{x}=\begin{bmatrix}5\\0\\-1\end{bmatrix}+t\begin{bmatrix}1\\1\\-4\end{bmatrix}$ 7. (a) $\cos^{-1}2/\sqrt{66}$ (b) $\cos^{-1}2/15$ 8. (b) $x+y+z=3$, $\mathbf{x}=\begin{bmatrix}1\\1\\1\end{bmatrix}+s\begin{bmatrix}1\\-1\\0\end{bmatrix}+t\begin{bmatrix}1\\0\\-1\end{bmatrix}$ 9. (a) (-11, 4, 6) (d) The entire line lies in the plane.

10. (a) $k=1$ (b) $k=-5/2$ (c) $k=-3$ 11. 12. (a) $x-2y+z=-1$ (b) 13 14. 15. (a) T (b) F (c) F (d) F

Section 5.5 1.(a)[-4,-10], [5, -2] (b) [-9/14, -9/7, 27/14],[23/14, 16/7, 29/14] (c) [1/3, 2/3, 1, 4/3],[2/3, 1/3, 0, -1/3] (d) [0, 7/26, -35/26], [3, 45/26, 9/26] 2. (a) $5\sqrt{2}/2$ (b) $8\sqrt{13}/13$ (c) $2/\sqrt{22}/11$ (d) $\sqrt{6}$ 3. (a) $6/\sqrt{17}$ (b) $1/\sqrt{11}$ (c) $13/\sqrt{6}$ (d) $3/2$ 4. (a) [-1/2, 3/2] (b) [1,-1/10,3/10] (c) [15/11, 16/11, 38/11] 5. (a) [4/11, 5/11, -5/11] (b) [2,2,0] (c) [9/7,11/7,2/7,-2/7]

Section 5.6 1. $-\mathbf{i}$ and $\mathbf{0}$ 2. (a) [5,-5,-10] (b) [-10,10,20] (c) 0 3. (a) -2i-5j-6k (b) 2j+2k (c) 4i+10j+12k 4. (a) $\sqrt{230}$ (b) $\sqrt{195}$ (c) 17 5. (a) [5,-3,-4] (b) [5,-1,-7] (c) [-4,2,-2] (d) [0,-4,4]

6. (a) (6,3) or (2,9) or (0,-3) (b) all the same area: 15 7.(a) (b) $\sqrt{1283}$ 8. 9. 10.(a) 20 (b) 41 (c) 17 11. 12. (a) a plane through rthe origin with normal $\mathbf{u}$. (b) the same plane as (a) but translated (c) a line, the span of $\mathbf{u}$. (c) a cylinder !!, with axis given by $\mathbf{u}$.

Section 5.7 1. (a) $3\sqrt{11}/2$ (b) $x+3y-z=7$ (c) $\mathbf{x}=\begin{bmatrix}1\\3\\3\end{bmatrix}+s\begin{bmatrix}1\\1\\4\end{bmatrix}+t\begin{bmatrix}2\\-1\\-1\end{bmatrix}$ (d)$\mathbf{x}=\begin{bmatrix}1\\3\\3\end{bmatrix}+t\begin{bmatrix}1\\1\\4\end{bmatrix}$ 2. (a) $\sqrt{59}$ (b) $x+7y-3z=11$ (c) $\mathbf{x}=\begin{bmatrix}0\\2\\1\end{bmatrix}+s\begin{bmatrix}1\\-1\\-2\end{bmatrix}+t\begin{bmatrix}5\\1\\4\end{bmatrix}$ (d) $\mathbf{x}=\begin{bmatrix}0\\2\\1\end{bmatrix}+t\begin{bmatrix}1\\-1\\-2\end{bmatrix}$ 3. (a) $5\sqrt{2}/2$ (b)$3\sqrt{6}/2$ (c) $2\sqrt{3}$ 4. (a) at [0,-1,-3] (b) $2\sqrt{3}/3$ (c) $\sqrt{182/3}$ (d) $\sqrt{8/7}$ 5. $3/\sqrt{5}$ 6. $6/\sqrt{91}$ 7. 3/5 8. (b) and (c)