PCA Module

Maths DDB — Cal III

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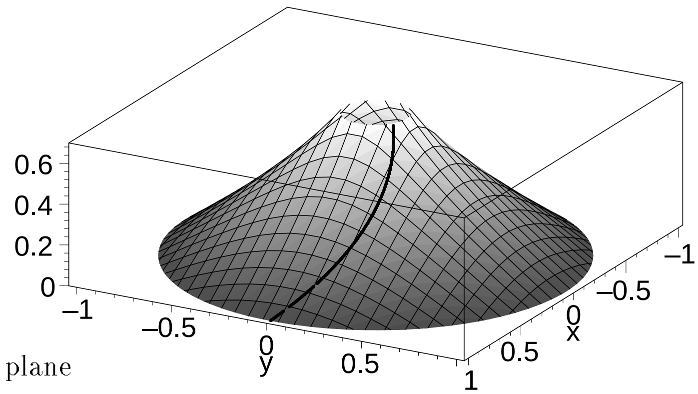
The Tower of Babel
A module assignment[1]



Pieter Bruegel the Elder painted the *Tower of Babel* in 1563. According to Genisis 11:1-9, Noah's descendants, under the direction of King Nimrod, assembled to build a tower reaching to Heaven. The equation of the curved side is

$z = 1 - \sqrt[3]{x^2 + y^2}$

for (x, y) in the ring-shaped region R in the xy plane

$$R = \{(x, y) \mid \left(\tfrac{1}{64}\right)^2 \leq x^2 + y^2 \leq 1\}$$

(x, y, z measured in km). The cylindrical part in the center of the tower was filled with rubble to the same height as the curved part.

King Nimrod decided to climb the curved side of the tower following the curve

$$\boldsymbol{r}(t) = \left(1 - \frac{4t}{9}\right)^3 \cos t \; \boldsymbol{i} + \left(1 - \frac{4t}{9}\right)^3 \sin t \; \boldsymbol{j} + \left(\frac{72t - 16t^2}{81}\right) \boldsymbol{k}$$

(t measured in hours), starting at $t = 0$, as shown by the thick black curve on the surface. At time $t = \frac{\pi}{2}$ Nimrod is at a point A described by the position vector

$$\overrightarrow{OA} = \left(1 - \frac{2\pi}{9}\right)^3 \boldsymbol{j} + \frac{(36 - 4\pi)\pi}{81} \boldsymbol{k}$$

Questions:

1. Write the equation of the tangent plane to the tower at A.
2. What is the direction of fastest ascent at the point A?
3. What is the directional derivative at the point A in the direction of the vector $\boldsymbol{i} + \boldsymbol{j}$?
4. Using a double integral in polar coordinates, compute the volume of the part of the tower which lies above the region R.
5. Find the whole volume of the tower, including the cylindrical part in the center.
6. Find the velocity vector $\boldsymbol{v}(t)$ and the acceleration vector $\boldsymbol{a}(t)$ of Nimrod's trajectory.
7. Write the equation of the tangent line to his trajectory at A.
8. What is his rate of ascent (rate of increase of the height as a function of t) at time $t = \frac{\pi}{2}$? Use the chain rule for multivariable functions.
9. Find his speed $v(t)$ and the arclength he travels on the tower from time $t = 0$ to $\frac{\pi}{2}$.

[1] Taken from Math 189-222B Final Exam, Apr. 2001, McGill University.