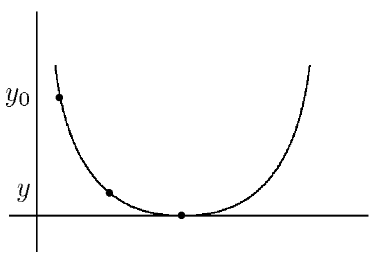
Some properties of the (inverted) cycloid
A module assignment

In this assignment you will have to show some physical properties of the inverted cycloid: we shall show that it is both a tautochrone ("same time") and a brachistochrone ("least time"). (Explain what a "tautochrone" and a "brachistochrone" are.) This means that you have to show that the time taken in moving from a point on the inverted cycloid to the bottom of the arch, moving only under the force g of gravity, is independant of the point chosen, and that to minimize the time taken in going from one point to the lower point, the path taken must be an inverted cycloid. The following set of "structured hints" will lead you to these conclusions.

Part I



We start with the simplification that $r = 1$, so the equations of the inverted cycloid are

$$x = \theta - \sin\theta\ , \quad y = 1 + \cos\theta$$

(These equations are somewhat different from the ones we derived in class, because of the fact that the cycloid has been inverted — **briefly** justify this variant of the equations.)

We imagine a body moving from point $y = y_0$ to $y = 0$; since $\frac{ds}{dt} = v$, show that the total time for this is

$$T = \int_{y=y_0}^{y=0} \frac{ds}{v}$$

Now, since energy is conserved (assume no friction as well), deduce $mgy_0 = mgy + \frac{1}{2}mv^2$ and so

$$v = \sqrt{2g(y_0 - y)}$$

Next, using $ds = \sqrt{dx^2 + dy^2}$, and the fact that dy is negative (see the graph), deduce (there are intermediate steps you need to fill in) that

$$ds \ = \ -\sqrt{\frac{2}{y}}\, dy$$

Putting this in the equation for T, show (again, there are missing steps you need to calculate)

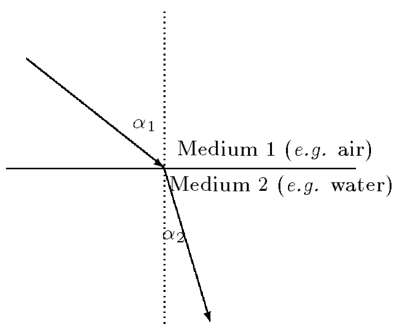
$$T = \frac{\pi}{\sqrt{g}}$$

(which doesn't depend on the initial position y_0).

Show that in the general case, where $x = r(\theta - \sin\theta)$ and $y = r(1 + \cos\theta)$, $(r > 0)$, the total time is $\frac{\pi\sqrt{r}}{\sqrt{g}}$ and so is still independant of the initial position. Conclude that the inverted cycloid is a tautochrone.

Part II

Next we shall show that the inverted cycloid minimizes the time taken in travelling from one point to another lower point. First recall that by Fermat's Principle, light travels through media along a path that minimizes the time taken: a consequence of this is *Snell's Law*, which states that the ratio $\frac{\sin\alpha}{v}$ is constant, say $= k$, where α is the angle of incidence, that is, the angle between a ray of light and the normal to the surface through which it passes (as illustrated), and v is the velocity at which the light travels through the medium.



$$\frac{\sin\alpha_1}{v_1} = \frac{\sin\alpha_2}{v_2} = k$$

Prove this (it was a Cal I exercise: *e.g.* Thomas, 5^{th} ed, p339, number 51).

Next, using the equation for v you derived on the previous page, together with the equation (show why this is true!)

$$\frac{dx}{ds} = \sin\alpha$$

conclude that

$$v\sqrt{1 + (y')^2} = k$$

This is a separable differential equation: solve it (it may help if you use the following substitution:

$$\sqrt{\frac{y}{k - y}} = \tan t$$

you will want to assume the curve goes through the origin — otherwise there will be an unwanted "+ C".) Show that the solution you get may be written:

$$\begin{cases} y = \frac{k}{2}(1 - \cos 2t) \\ x = \frac{k}{2}(2t - \sin 2t) \end{cases}$$

If you don't recognise this as our cycloid, substitute $\theta = 2t$ and $r = \frac{k}{2}$. Justify that this really is an inverted cycloid.