Envelopes
A module assignment

Frequently one has a family of curves which, when all plotted on the same axes, seem to define another curve implicitly. For example, you have probably seen string and nail constructions which seem to describe a circle (or other "round" figure) in terms of a large number of straight lines. This implicit curve is called the *envelope* of the family; our job is to develop a method of finding its equation, given the equations of the family of curves.

The context

We start with a family of curves in the x, y plane, given by an equation $f(x, y, t) = 0$ involving a parameter t.

As a "running example", we shall take the family $\frac{x}{k} + ky = 2$ where t is k, and $f(x, y, k) = \frac{x}{k} + ky - 2$. Show that this family consists of straight lines with x intercepts $2k$ and y intercepts $\frac{2}{k}$. Graph the lines corresponding to the values $k = \pm 2, \pm 1.5, \pm 1, \pm .5$. What happens if $k = 0$?

We say that a curve $\mathcal{C}$ is the *envelope* of the family $f(x, y, t) = 0$ if for each value of t, the curve $f(x, y, t) = 0$ is tangent to $\mathcal{C}$ at some point $P_t(x_t, y_t)$ depending on t.

In the case of the family $\frac{x}{k} + ky = 2$ draw a sketch of the hyperbola $xy = 1$ on the same graph as you drew above with the family of lines, and verify that it seems to be the envelope of the family. (We shall check this equation again later.)

Determining the equation of the envelope

To determine the equation of $\mathcal{C}$, we shall assume that f has continuous first partial derivatives, and that $\mathcal{C}$ is smooth.

Notation: Let's denote the point P_t where the curve $f(x, y, t) = 0$ is tangent to $\mathcal{C}$ by $x_t = g(t)$ and $y_t = h(t)$. Then the equations $x = g(t), y = h(t)$ are parametric equations of $\mathcal{C}$. Since P_t lies on $f(x, y, t) = 0$, we know that $f(g(t), h(t), t) = 0$ for all t. Take the derivative with respect to t, and show that

$$\frac{\partial f}{\partial x}\frac{dg}{dt} + \frac{\partial f}{\partial y}\frac{dh}{dt} + \frac{\partial f}{\partial t} = 0$$

(when evaluated at P_t). Suppose $\frac{\partial f}{\partial y} \neq 0$. (Is this a reasonable assumption? What if it isn't true?) Then, we may suppose (justify!) that for any t, $f(x, y, t) = 0$ represents a function $y = y(x)$, and show that (at P_t) for this function,

$$\frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}$$

(Hint: implicit differentiation.)

Finally, since the envelope $\mathcal{C}$ is tangent to $f(x, y, t) = 0$

$$\frac{\partial f}{\partial x}\frac{dg}{dt} + \frac{\partial f}{\partial y}\frac{dh}{dt} = 0$$

(Hint: the envelope has slope $\frac{dh}{dt}/\frac{dg}{dt}$.)

Conclude that at all points of $\mathcal{C}$, $\frac{\partial f}{\partial t} = 0$. Hence conclude that the system of equations which determines $\mathcal{C}$ is

$$\begin{cases} f(x,y,t) &= 0 \\ \frac{\partial f}{\partial t}(x,y,t) &= 0 \end{cases}$$

(Eliminating the parameter t can give the equation in x, y alone.)

In the case of the family $\frac{x}{k} + ky = 2$ *show that the system above is*

$$\begin{cases} \frac{x}{k} + ky &= 2 \\ -\frac{x}{k^2} + y &= 0 \end{cases}$$

which has solution $x = k, y = \frac{1}{k}$, i.e. $xy = 1$, *as we seemed to have already established graphically.*

Example: The evolute of an ellipse

An ellipse may be given by the equations $x = a\cos t,\ \ y = b\sin t$, for some constants a, b. Suppose at each point P_t we draw a normal line N_t to the ellipse; what is the envelope of these normals?

To solve this, first we find the equation $f(x,y,t) = 0$ for the normal line N_t. Show that the slope of N_t is $\frac{a\sin t}{b\cos t}$. (Hint: find the slope of the tangent at P_t, and then remember "negative reciprocals".) Then (using the standard method for finding the equation of a straight line through a given point with a given slope) show that the normal N_t has the equation

$$y - b\sin t - \frac{ax\sin t}{b\cos t} + \frac{a^2}{b}\sin t = 0$$

(Hint: N_t goes through (x, y) given by the parametric equations above with the slope above.) This is the $f(x,y,t) = 0$ we were looking for. Now, show that the other condition $\frac{\partial f}{\partial t}(x,y,t) = 0$ becomes the equation

$$-b\cos t - \frac{ax}{b\cos^2 t} + \frac{a^2}{b}\cos t = 0$$

Solve this system of equations to show that

$$x = \frac{a^2 - b^2}{a}\cos^3 t \quad \text{and} \quad y = \frac{b^2 - a^2}{b}\sin^3 t$$

Remark: This envelope also describes the centers of the osculating circles for the ellipse; such a curve is called the evolute *of the original curve.*

(On my web page, you can find a graph of the case when $a = 2,\ b = 1$.)